M606 Generating Functions – Exponential

Note Title 2/14/2007

Defn

$f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ means the series f is the exponential generating function of the sequence $\{a_n\}_0^\infty$; ie, that

$$f = \sum_{n\geq 0} \frac{a_n}{n!} x^n$$

Note the derivative

$$f' = \sum_{n\geq 1} \frac{n a_n}{n!} x^{n-1} = \sum_{n\geq 1} \frac{a_n}{(n-1)!} x^{n-1}$$

$$= \sum_{n\geq 0} \frac{a_{n+1}}{n!} x^n$$

Rule A' If $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$, then

$$D^h f \overset{egf}{\longleftrightarrow} \{a_{n+h}\}_0^\infty$$

Ex Fibonacci. Let $\{F_n\}_0^\infty$ be Fibonacci (ie., $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ $\forall n \geq 0$) and let f be its egf.

$$f'' = f' + f$$

Now we solve the diff eq

$$f'' - f' - f = 0$$

Characteristic equation:

$$x^2 - x - 1 = 0$$

Roots: $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

$$f(x) = Ae^{r_1 x} + Be^{r_2 x}$$
$$f(0) = 1$$
$$f'(0) = 1$$
$$f'(x) = Ar_1 e^{r_1 x} + Br_2 e^{r_2 x}$$

$$\begin{cases} 1 = A + B \\ 1 = Ar_1 + Br_2 \end{cases}$$

$$\begin{cases} 1 = A + B \\ 1 = Ar_1 + (1-A)r_2 \end{cases}$$

$$\begin{cases} 1 - r_2 = A(r_1 - r_2) \\ B = 1 - A \end{cases}$$

$$\begin{cases} \left(1-\frac{1-\sqrt5}{2}\right) = A\left(\frac{1+\sqrt5}{2}-\frac{1-\sqrt5}{2}\right) \\ B = 1-A \end{cases}$$

$$\begin{cases} \frac{1+\sqrt5}{2} = A\sqrt5 \\ B = 1-A \end{cases}$$

$$A = \frac{r_1}{\sqrt5} \qquad B = 1-\frac{r_1}{\sqrt5} = \frac{2\sqrt5-(1+\sqrt5)}{2\sqrt5} = \frac{\sqrt5-1}{2\sqrt5} = -\frac{r_2}{\sqrt5}$$

$$f = \frac{r_1}{\sqrt5}e^{r_1x} - \frac{r_2}{\sqrt5}e^{r_2x}$$

$$= \frac{r_1}{\sqrt5}\sum_{n\ge0}\frac{(r_1x)^n}{n!} - \frac{r_2}{\sqrt5}\sum_{n\ge0}\frac{(r_2x)^n}{n!}$$

$$= \sum_{n\ge0}\left(\frac{1}{\sqrt5}r_1^{n+1} - \frac{1}{\sqrt5}r_2^{n+1}\right)\frac{x^n}{n!}$$

Rule B' If $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ and P is a polynomial, then

$$\left(P(xD)\right) f \overset{egf}{\longleftrightarrow} \{P(n)\,a_n\}_0^\infty$$

Proof

$$(xD)f = x\sum_{n\ge1}\frac{a_n}{(n-1)!}x^{n-1} = \sum_{n\ge1}\frac{a_n}{(n-1)!}x^n$$

$$= \sum_{n\ge1} n\,a_n\frac{x^n}{n!}$$

Multiply two series

$$\left(\sum_{m\geq 0} a_m \frac{x^m}{m!}\right)\left(\sum_{k\geq 0} b_k \frac{x^k}{k!}\right)$$

$$= \sum_{m,k\geq 0} \frac{a_m b_k}{m!\,k!} x^{m+k}$$

$$= \sum_{n\geq 0}\left(\sum_{m+k=n} \frac{a_m b_k}{m!\,k!}\right) x^n$$

$$= \sum_{n\geq 0}\left(\sum_{m+k=n} \frac{n!}{m!\,k!} a_m b_k\right)\frac{x^n}{n!}$$

$$= \sum_{n\geq 0}\left(\sum_{m} \binom{n}{m} a_m b_{n-m}\right)\frac{x^n}{n!}$$

<u>Rule C'</u> If $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ and $g \overset{egf}{\longleftrightarrow} \{b_n\}_0^\infty$ then

$$fg \overset{egf}{\longleftrightarrow} \left\{\sum_m \binom{n}{m} a_m b_{n-m}\right\}$$

<u>Example</u> Bell numbers

$$B(n+1) = \sum_k \binom{n}{k} B(k) \qquad B(0)=1$$

$$\{B(n+1)\}_0^\infty \xleftrightarrow{\text{egf}} f'$$

$$\left\{\sum_k \binom{n}{k} B(k)\right\} \xleftrightarrow{\text{egf}} fg, \quad \text{where } g \xleftrightarrow{\text{egf}} \{1\}_0^\infty$$

$$g = e^x$$

$$\boxed{f' = f e^x \quad ; \quad f(0) = 1}$$

$$\frac{f'}{f} = e^x$$

$$(\ln f)' = e^x$$

$$\ln f = e^x + C$$

$$\begin{cases} f = \exp\{e^x + C\} \\ 1 = \exp\{1 + C\} \iff C = -1 \end{cases}$$

Thus, $f = \exp\{e^x - 1\}$

<u>Example</u> Derangements.

$$n! = \sum_k \binom{n}{k} D_{n-k} \qquad \forall n \geq 0$$

$$= \sum_k \binom{n}{n-k} D_k = \sum_k \binom{n}{k} D_k$$

$$\{n!\} \xleftrightarrow{egf} \frac{1}{1-x}$$

$$\{D_k\} \xleftrightarrow{egf} D(x) \qquad \text{(define)}$$

$$\{1\} \xleftrightarrow{egf} e^x$$

$$\left\{\sum_k \binom{n}{k} D_k\right\} \xleftrightarrow{egf} e^x D(x) \qquad \text{(Rule 3')}$$

$$\frac{1}{1-x} = e^x D(x)$$

$$D(x) = \frac{e^{-x}}{1-x}$$

$$e^{-x} \xleftrightarrow{egf} \{(-1)^n\}_n$$

$$\frac{1}{1-x} \xleftrightarrow{egf} \{n!\}_n$$

By Rule 3',

$$\left\{\frac{e^{-x}}{1-x}\right\} \overset{egf}{\longleftrightarrow} \left\{\sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!\right\}$$

$$\longleftrightarrow \left\{\sum_{k=0}^{n} \frac{n!}{k!} (-1)^k\right\}$$

So, $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

<u>Rule D'</u> If $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$, then

$$f^k \overset{egf}{\longleftrightarrow} \left\{\sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, \ldots, r_k} a_{r_1} a_{r_2} \cdots a_{r_k}\right\}_{n=0}^{\infty}$$