M606 Generating Functions – Dirichlet

Note Title 3/20/2009

Dirichlet power series

Let $\{a_n\}_{n\geq 0}$ be a sequence. The Dirichlet generating function of $\{a_n\}_{n\geq 0}$ is

$$f(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

Denoted $\{a_n\}_{n\geq 1} \xleftrightarrow{\text{Dir}} f$.

Useful in number theory. We won't deal with it.

$$\{1\} \xleftrightarrow{\text{ops}} \frac{1}{1-x}$$

$$\{1\} \xleftrightarrow{\text{egf}} e^x$$

$$\{1\} \xleftrightarrow{\text{Dir}} \zeta(s),$$ the Riemann zeta function.

Möbius inversion and cyclotomic polynomials are conveniently handled.

Rule C''

If $f \xleftrightarrow{Dir} \{a_n\}_1^\infty$ and $g \xleftrightarrow{Dir} \{b_n\}_1^\infty$

then $fg \xleftrightarrow{Dir} \left\{ \sum_{d|n} a_d b_{n/d} \right\}$

Rule D''

If $f \xleftrightarrow{Dir} \{a_n\}_1^\infty$, then

$$f^k \xleftrightarrow{Dir} \left\{ \sum_{(d_1,\dots,d_k)} a_{d_1} a_{d_2} \cdots a_{d_k} \right\}_1^\infty$$

Proof of Rule C''

$$fg = (a_1 + a_2 2^{-s} + a_3 3^{-s} + \dots)(b_1 + b_2 2^{-s} + b_3 3^{-s} + \dots)$$

$$= (a_1 b_1) + (a_1 b_2 + a_2 b_1) 2^{-s} + (a_1 b_3 + a_3 b_1) 3^{-s}$$

$$+ (a_1 b_4 + a_2 b_2 + a_4 b_1) 4^{-s} + \dots$$

Example $\zeta^2(s)$

$$\zeta \xleftrightarrow{Dir} \{1\}_1^\infty \Longrightarrow \zeta^2 \xleftrightarrow{Dir} \left\{ \sum_{d|n} 1 \right\}_1^\infty$$

$$\xleftrightarrow{Dir} \{ \#\text{divisors of } n \}_1^\infty$$

$$\zeta^k \overset{\text{Dir}}{\longleftrightarrow} \left\{ \#\{(d_1,\dots,d_k) \mid n = d_1 \cdots d_k\} \right\}_1^\infty$$

Defn

A function is a number theoretic function if its domain is positive integers. A number theoretic function is multiplicative if $f(mn) = f(m)f(n)$ for all relatively prime m and n.

Clearly, if $n = p_1^{a_1} \cdots p_r^{a_r}$ then f multiplicative means $f(n) = f(p_1^{a_1}) \cdots f(p_r^{a_r})$.

Theorem Let f be a multiplicative, number-theoretic function. Then, formally,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left\{ 1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots \right\}$$

Proof Ex. Similar to proof of Rule C''.

Example 1 $f(n)=1 \quad \forall n \geq 1$

$$\sum_{1}^{\infty} \frac{1}{n^s} = \prod_p \left\{ 1 + p^{-s} + p^{-2s} + \cdots \right\}$$

$$= \prod_p \frac{1}{1-p^{-s}} = \left(\prod_p \left(1-p^{-s}\right) \right)^{-1}$$

Example 2 $\mu(n)$

Note that $\mu(n) = \begin{cases} 0 & \text{if } n \text{ not squarefree} \\ (-1)^r & \text{if } n = p_1 \cdots p_r \end{cases}$

is multiplicative.

$$\sum_{1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left\{ 1 + \mu(p) p^{-s} + \mu(p^2) p^{-2s} + \mu(p^3) p^{-3s} + \cdots \right\}$$

$$= \prod_p \left(1 - p^{-s}\right) = \frac{1}{\zeta(s)}$$

i.e. $\frac{1}{\zeta(s)} \overset{\text{Dir}}{\longleftrightarrow} \{\mu(n)\}_1^{\infty}$

Möbius inversion

Let $a_n = \sum_{d|n} b_d \qquad n \geq 1$

and $A(s) \overset{Dir}{\longleftrightarrow} \{a_n\}_1^\infty \quad B(s) \overset{Dir}{\longleftrightarrow} \{b_n\}_1^\infty$

By Rule C'',

$$A(s) = B(s)\,\zeta(s)$$

So,

$$B(s) = A(s)\frac{1}{\zeta(s)} \overset{Dir}{\longleftrightarrow} \left\{\sum_{d|n} a_d \mu\left(\frac{n}{d}\right)\right\}_1^\infty$$

$$b_n = \sum_{d|n} a_d \mu\left(\frac{n}{d}\right)$$

Theorem [Möbius inversion]

$$a_n = \sum_{d|n} b_d \quad \forall n \iff b_n = \sum_{d|n} a_d \mu\left(\frac{n}{d}\right)$$

<u>Example</u> 0-1 strings

A <u>primitive string</u> is one that is not the concatenation of smaller substrings

not primitive : 100100100
primitive : 1101

Let $f(n)$ be the number of 0-1 strings of length n.

$$2^n = \sum_{d|n} f(d)$$

By Möbius inversion,

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d.$$