M606 Generating Functions - Analysis

Note Title 2/14/2007

Defn

If f is an ordinary power series, then the radius of convergence (RoC) is the largest R such that $f(z)$ converges for all $|z|<R$ but diverges for some $|z| \geq R$

If f diverges for some z_0, then z_0 is said to be a singularity.

Theorem 1 Let f be an ops with ROC R. Then

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \quad \text{where } \frac{1}{\infty} = 0, \ \frac{1}{0} = \infty$$

Proof
In generatingfunctionology or a complex analysis book.

Theorem 2 If $\sum_{n\geq 0} a_n z^n$ has RoC R, then f is analytic in $|z|<R$

and if $R < \infty$ then $\sum_{n \geq 0} a_n z^n$ has at least one singularity for $|z| = R$.

Proof Ditto

Ex $\quad \sum z^n \qquad R = 1$

Ex $\quad f(z) = \frac{1}{2 - e^x} \qquad R = \ln 2$

Ex $\quad f(z) = \frac{z}{e^z - 1}$

Singularities $\quad e^z = 1 \Leftrightarrow z = (2\pi i)k$

But $\quad \lim_{z \to 0} \frac{z}{e^z - 1} \overset{L'H}{=} \lim_{z \to 0} \frac{1}{e^z} = 1$

So, the closest singularities to the origin are $\pm 2\pi i \Rightarrow R = 2\pi$

Thus, $\quad \overline{\lim_{n \to \infty}} |a_n|^{\frac{1}{n}} = \frac{1}{2\pi}$

Therefore, $\forall \varepsilon \ \exists N$ s.t. $\forall n \geq N$

$$|a_n|^{\frac{1}{n}} < \frac{1}{2\pi} + \varepsilon$$

$$|a_n| < \left(\frac{1}{2\pi} + \varepsilon\right)^n$$

For infinitely many n,

$$|a_n|^{\frac{1}{n}} > \frac{1}{2\pi} - \varepsilon$$

$$|a_n| > \left(\frac{1}{2\pi} - \varepsilon\right)^n$$

Theorem 3 Let $f(z) = \sum a_n z^n$ be analytic in some region containing the origin, let a singularity of $f(z)$ of smallest modulus be at a point $z_0 \neq 0$ and let $\varepsilon > 0$ be given. Then there exists N such that for all $n > N$ we have

$$|a_n| < \left(\frac{1}{|z_0|} + \varepsilon\right)^n$$

Further, for infinitely many n, we have

$$|a_n| > \left(\frac{1}{|z_0|} - \varepsilon\right)^n.$$

Proof Complex analysis.

Cauchy's formula

If $f(z) = \sum a_n z^n$, then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z^{n+1}} \qquad n \in \mathbb{N}$$

where C encloses the origin and lies entirely inside a region in which f is analytic.

Cauchy's inequality

$$|a_n| \leq \frac{M(r)}{r^n}$$

which holds $\forall n \geq 0$, and $0 < r < R$, where $R =$ ROC and

$$M(r) = \max_{|z| \leq r} |f(z)| = \max_{|z| = r} |f(z)| .$$

Proof of CI:

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z^{n+1}} \qquad \begin{aligned} z &= re^{i\theta} \\ dz &= ire^{i\theta}d\theta \end{aligned}$$

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta})^{n+1}} \, ire^{i\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta})^n} \, d\theta$$

$$|a_n| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta})|}{|re^{i\theta}|^n} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta})|}{r^n} \, d\theta$$

$$\le \frac{M(r)}{2\pi r^n} \int_0^{2\pi} d\theta = \frac{M(r)}{r^n}. \quad \square$$

Subsequences:

$\frac{f(x)+f(-x)}{2}$ has just coefficients of even powers of x

$\frac{f(x)-f(-x)}{2}$ has just coefficients of odd powers of x

Example: What is $g(x) = \sum_{n \ge 0} \frac{x^{3n}}{(3n)!}$?

$$\frac{1^n + (-1)^n}{2} = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

Let $\{1, w_1, w_2\}$ be the cube roots of unity

$$\frac{1^n + \omega_1^n + \omega_2^n}{3} = \begin{cases} 1 & \text{if } 3 \mid n \\ 0 & \text{else} \end{cases}$$

Let $f(x) = \sum_{n \geq 0} \frac{x^n}{n!}$

$$\frac{f(x) + f(\omega_1 x) + f(\omega_2 x)}{3} = \sum_{n : 3 \mid n} \frac{x^n}{n!} = \sum_{n \geq 0} \frac{x^{3n}}{(3n)!}$$

$$\frac{e^x + e^{\omega_1 x} + e^{\omega_2 x}}{3} = \sum_{n \geq 0} \frac{x^{3n}}{(3n)!}$$

$$\begin{aligned} \omega_1 &= e^{2\pi i/3} \\ &= \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \\ &= \frac{-1}{2} + i \frac{\sqrt{3}}{2} \end{aligned} \qquad \begin{aligned} \omega_2 &= e^{4\pi i/3} \\ &= \cos\left(\frac{2\pi}{3}\right) - i \sin\left(\frac{2\pi}{3}\right) \\ &= \frac{-1}{2} - i \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \sum_{n \geq 0} \frac{x^{3n}}{(3n)!} &= \frac{1}{3}\left[e^x + e^{\frac{-1}{2}x + i\frac{\sqrt{3}}{2}x} + e^{\frac{-1}{2}x - i\frac{\sqrt{3}}{2}x}\right] \\ &= \frac{1}{3}\left[e^x + e^{-x/2}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) + i \sin\left(\frac{\sqrt{3}}{2}x\right)\right)\right. \\ &\qquad \left. + e^{-x/2}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) - i \sin\left(\frac{\sqrt{3}}{2}x\right)\right)\right] \end{aligned}$$

$$\sum_{n\geq 0} \frac{x^{3n}}{(3n)!} = \frac{1}{3}\left[e^{x} + 2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

<u>Proposition</u> $\displaystyle \frac{1}{r}\sum_{\omega^r=1} \omega^n = \begin{cases} 1 & \text{if } r \mid n; \\ 0 & \text{else.} \end{cases}$

<u>Proof</u> LHS is

$$\frac{1}{r}\sum_{j=0}^{r-1} \exp\left\{\frac{2\pi i n}{r} j\right\} \qquad \square$$

Useful Power Series:

See pp. 52-56 in generatingfunctionology, second edition for a number of useful power series.