M606 Extremal — Erdős Ko Rado

Note Title 4/24/2007

<u>Theorem.</u> [Erdős–Ko–Rado, 1967)

Let $\mathcal{H}$ be an r-uniform intersecting hypergraph. Then, for $r \le n/2$

$$|\mathcal{H}| \le \binom{n-1}{r-1}$$

with equality for $r < n/2$ iff $\mathcal{H}$ fixes some x.

<u>Proof</u> (Daykin '74)

Choose an $\mathcal{A}$ for which

$$|\mathcal{A}| = \binom{n-1}{r-1}$$

$\partial^t \mathcal{A}$ is the t^{th} shadow of $\mathcal{A}$

Let $\mathcal{B} = \mathcal{A}^c = \{X \setminus A : A \in \mathcal{A}\} \subseteq \binom{X}{n-r}$

$A \in \mathcal{A},\ B \in \mathcal{B} \Rightarrow A \not\subset B$

$$\Rightarrow (\partial^{n-2r}\mathcal{B}) \cap \mathcal{A} = \emptyset$$

$$|\mathcal{A}| = |\mathcal{B}| = \binom{n-1}{r-1} = \binom{n-1}{n-r}$$

$$|\partial \mathcal{B}| \geq \binom{n-1}{n-r-1}$$ with equality iff $\mathcal{B} = \binom{Y}{n-r}$ for some $Y \subseteq X$, $|Y| = n-1$

$\vdots$

$$|\partial^t \mathcal{B}| \geq \binom{n-1}{n-r-t}$$ w/ eq. iff $\mathcal{B} = \binom{Y}{n-r}$ for some $Y \subseteq X$ (again, $|Y| = n-1$.)

$$|\partial^{n-2r} \mathcal{B}| \geq \binom{n-1}{n-r-(n-2r)} = \binom{n-1}{r}$$

Since $\partial^{n-2r}\mathcal{B} \,\dot\cup\, \mathcal{A} \subseteq \binom{X}{r}$,

$$|\partial^{n-2r}\mathcal{B}| + |\mathcal{A}| \leq \binom{n}{r} \Leftrightarrow$$

$$\binom{n-1}{r} + \binom{n-1}{r-1} \leq |\partial^{n-2r}\mathcal{B}| + |\mathcal{A}| \leq \binom{n}{r}$$

By Pascal's Δ, we have equality.

Thus, $\mathcal{B} = \binom{Y}{r}$ for some $Y \subseteq X$, $|Y| = n-1$.

Let $x = X \setminus Y$. Since $\mathcal{B} = \mathcal{A}^c$, it must be that $x \in A$, $\forall A \in \mathcal{A}$.

So, $|\mathcal{A}| = \binom{n-1}{r-1} \Rightarrow \mathcal{A}$ fixes x.
Since an intersecting hypergraph that fixes x is maximal, there is no intersecting hypergraph with more edges.

Open Question

<u>Conjecture</u> [Simonyi, '89]

"Hourglass conjecture"

Given $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ such that

- $A \cup B = A' \cup B' \Rightarrow B = B'$ and
- $A \cap B = A' \cap B' \Rightarrow A = A'$

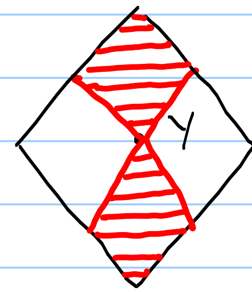
Then $|\mathcal{A}||\mathcal{B}| \leq 2^n$

Example

Let $Y \in 2^{[n]}$.

Let $\mathcal{B} = U(Y)$

$\mathcal{A} = D(Y)$



Notes taken from <u>Combinatorics</u> by Bela Bollobás.