Note Title 1/22/2007

M606 Counting : The Twelvefold Way

Let $f: N \to X$, $n = |N|$, $x = |X|$

We want to count the number of functions

- arbitrary
- injective
- surjective.

We also have to determine whether the items in N or X are distinguishable.

<u>Eg.</u>

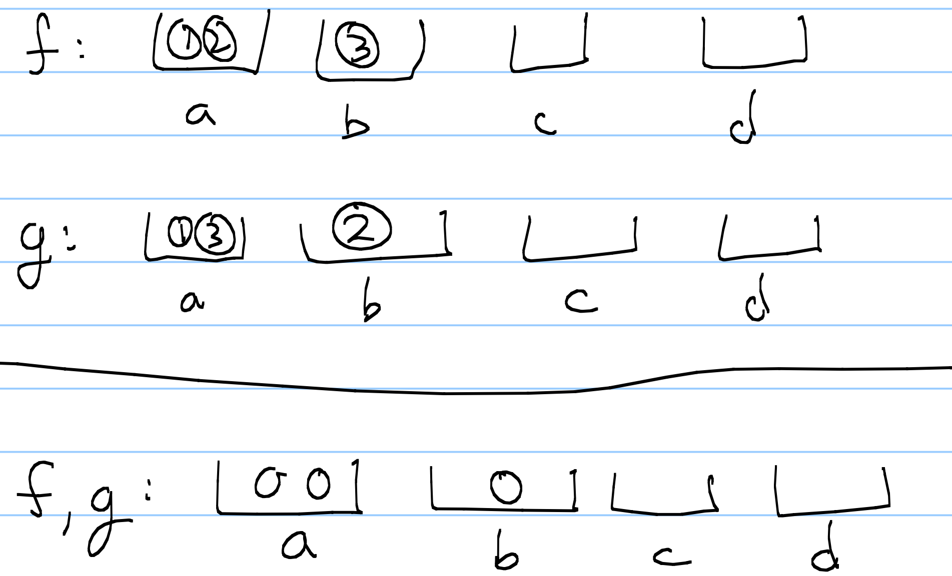
$f(1) = f(2) = a \quad f(3) = b$

$g(1) = g(3) = a \quad g(2) = b$

$h(1) = h(2) = b \quad h(3) = d$

$i(2) = i(3) = b \quad i(1) = c$

Diagrams



The Twelvefold Way:

N	X	Any f	Injective f	Surjective f
dist	dist	1	2	3
indist	dist	4	5	6
dist	indist	7	8	9
indist	indist	10	11	12

Easy: 1 $= x^n$

2 $= (x)_n$

(4) We are only interested in how many balls are in each box $b_1, b_2, \dots, b_x$

$\#$ of n-element multisets of X

$$= \left(\!\binom{x}{n}\!\right) = \binom{x+n-1}{n}$$

(5) Each box contains ≤ 1 ball

They're subsets, so it's $\binom{x}{n}$.

(6) Balls are indistinguishable, boxes are. So, put a ball in each box and count the number of remaining functions as in (4). So, $\left(\!\binom{x}{n-x}\!\right)$.

(12) This is a partition of n into exactly x parts.

$$p_x(n)$$

(10) This is a partition of n into at most x parts

$$p_1(n) + p_2(n) + \cdots + p_x(n)$$

(8) Put each ball into a different indist. box. There is only one possible way to do this, unless there are too few boxes.
$$\begin{cases} 1, & n \le x \\ 0, & n > x \end{cases}$$

(11) This is the same as (8)
$$\begin{cases} 1, & n \le x \\ 0, & n > x \end{cases}$$

(9) Balls are distinguishable, boxes are not and boxes are not empty.

So, these functions correspond to a partition of $[n]$ into x nonempty pieces.

This number is the Stirling number of the second kind $S(n,x)$.

More on these later

(7) $S(n,1) + S(n,2) + \cdots + S(n,x)$

(3) $x!\, S(n,x)$

N	X	Any f	Inj. f	Surj. f
dist	dist	x^n	$(x)_n$	$x!\,S(n,x)$
indist	dist	$\left(\!\binom{x}{n}\!\right)$	$\binom{x}{n}$	$\left(\!\binom{x}{n-x}\!\right)$
dist	indist	$S(n,1)+S(n,2)+\dots+S(n,x)$	$\begin{cases}1, & n\le x\\ 0, & n>x\end{cases}$	$S(n,x)$
indist	indist	$P_1(n)+P_2(n)+\dots+P_x(n)$	$\begin{cases}1, & n\le x\\ 0, & n>x\end{cases}$	$P_x(n)$