Homework 4

Spring 2009 M606:
Enumerative Combinatorics and Partially Ordered Sets

Due Apr. 7, 2009, assigned Mar. 31, 2009

You may ask me any questions in office hours.
LaTeX or other typed solutions are very strongly preferred.

1 Do all four (4) of the following:

PROBLEM 1 *We will use Möbius inversion as defined in the notes "Generating Functions – Dirichet". Let φ be the Euler phi function. That is, $\varphi(n)$ counts the number of integers at most n which are relatively prime to n ($\varphi(1) = 1$). Let μ be the Möbius function. That is, $\mu(n)$ is zero if n is divisible by the square of some prime and is $(-1)^r$ if n is the product of r distinct primes ($\mu(1) = 1$).*

a. *Prove that, for all $n \geq 1$,*

$$n = \sum_{d|n} \varphi(d).$$

*(**Hint:** How many $i \in \{1, \ldots, n\}$ have $\gcd(i, n) = d$?)*

b. *Use Möbius inversion to prove that, for all $n \geq 1$,*

$$\varphi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \cdot d.$$

PROBLEM 2 *Let $(\mathcal{P}, \leq_{\mathcal{P}})$ and $(\mathcal{Q}, \leq_{\mathcal{Q}})$ be posets.*

a. *Prove that there is a poset, which we denote $(\mathcal{P} \times \mathcal{Q}, \leq_{\mathcal{P}\times\mathcal{Q}})$ such that the elements are the members of the Cartesian product of $\mathcal{P}$ and $\mathcal{Q}$ and, for $p, p' \in \mathcal{P}$ and $q, q' \in \mathcal{Q}$ it is the case that*

$$(p, q) \leq_{\mathcal{P}\times\mathcal{Q}} (p', q') \qquad \textit{iff} \qquad \textit{both } p \leq_{\mathcal{P}} p' \textit{ and } q \leq_{\mathcal{Q}} q'.$$

b. *Prove that the Boolean lattice (that is, the poset of subsets of $[n]$, ordered by set inclusion) is isomorphic to $\mathbf{2}^n$, the product of n copies of a chain of 2 elements.*

In the next two problems, we will derive the proof that Dilworth's Theorem and the König-Egerváry theorems are equivalent. The proof is due to Fulkerson [1956]. Feel free to discuss this with your classmates, but you must turn in your own writeup. First, we summarize the theorems:

Theorem 1 (Dilworth, 1950) *If $\mathcal{P}$ is a finite poset, then the maximum size of an antichain in $\mathcal{P}$ equals the minimum number of chains needed to cover the elements of $\mathcal{P}$.*

Theorem 2 (König, 1931 and Egerváry, 1931) *A **vertex cover** of a graph G is a set of vertices such that each edge is incident to at least one of these vertices.*

In a bipartite graph G, the size of the largest matching is equal to the size of the smallest vertex cover.

PROBLEM 3 ***Dilworth implies König-Egerváry*** *To prove this, let us be given a bipartite graph $G = (A, B; E)$ on n vertices and construct a bipartite poset $\mathcal{P}$ with A the maximal elements and B the minimal elements and $a \geq b$ iff $a \sim b$ in G. This is a **bipartite poset**.*

(i). Show that if $\mathcal{P}$ has a chain cover of size $n - k$, then G has a matching of size k.

(ii). Show that if $\mathcal{P}$ has an antichain of size $n - k$, then it has a vertex cover of size k.

(iii). Use (i) and (ii) to prove that the size of the largest matching in G is equal to the size of the smallest vertex cover.

PROBLEM 4 ***König-Egerváry implies Dilworth*** *To prove this, let us be given a poset $\mathcal{P}$ on n elements and construct a bipartite graph $G = (A^+, A^-; E)$ on $2n$ vertices in which we have two copies of each $x \in \mathcal{P}$. The first copy is $x^+ \in A^+$ and the second copy is $x^- \in A^-$. The edge set is $\{x^- y^+ : x <_{\mathcal{P}} y\}$. This graph G is called the **split of** $\mathcal{P}$, denoted $S(\mathcal{P})$.*

*(i). Show that if G has a matching of size k, then there is a cover with $n - k$ chains. **Hint:** If $\{x_1^-, x_2^+\}$ and $\{x_2^-, x_3^+\}$ then the corresponding chain is $x_1 < x_2 < x_3$. Why can't $x_1 = x_3$?*

(ii). (a) Show that if G has a vertex cover C then $A = \{x \in P : x^-, x^+ \notin C\}$ is an antichain in P.

(b) Show that if C is minimal that it is not possible for both of $\{y^+, y^-\}$ to be in C. It is possible for C to be have vertices in both A^+ and A^-, but we can regard it as a subset of P.

(c) Show that if G has a minimum vertex cover of size $n-k$, then $\mathcal{P}$ has an antichain of size k.

(iii). Use (i) and (ii) to prove that the size of the smallest chain cover in $\mathcal{P}$ is equal to the size of the largest antichain.

Comments on the equivalence of theorems

In fact, there are (at least) nine theorems that can be quickly proven to be equivalent to each other. Not only Dilworth's theorem and the König-Egerváry theorem but also the following seven:

Theorem 3 (Menger, 1929) *A* vw***-separating set*** S *in a graph* G *is a subset of the vertices so that there is no path from* v *to* w *in* $G \setminus S$.

The maximum number of vertex-disjoint paths connecting two distinct non-adjacent vertices v *and* w *is equal to the minimum number of vertices in a* vw*-separating set.*

Theorem 4 (König's theorem for matrices, 1931) *The **term rank** of a* $(0,1)$*-matrix is the largest number of 1s that can be chosen so that no 2 selected 1s are in the same row or column.*

A set S *of rows and columns is a **cover** of a* $(0,1)$*-matrix if the matrix has no 1s not in* S.

The term rank of a $(0,1)$*-matrix is the size of its smallest cover.*

Theorem 5 (P. Hall, 1935) *Let* $S = \{S_1, \ldots, S_n\}$ *be a family of subsets of a ground set* X*. Then, a **system of distinct representatives (SDR)** for* S *is a sequence of distinct elements* $\{x_1, \ldots, x_n\}$ *of* X *such that* $x_i \in S_i$, $1 \leq i \leq n$.

The family S *has an SDR iff the union of any* k *members of* S *contains at least* k *elements.*

Theorem 6 (Birkhoff, 1946 and von Neumann, 1953) *A matrix with real nonnegative entries is **doubly stochastic** if the sum of the entries in any row and any column equals one. A **permutation matrix** is a doubly stochastic* $(0,1)$*-matrix. A matrix* $\mathbf{A}$ *is a **convex combination** of matrices* $\mathbf{A}_1, \ldots, \mathbf{A}_s$ *if there exist nonnegative reals* $\lambda_1, \ldots, \lambda_s$ *such that* $\sum_{i=1}^{s} \lambda_i = 1$ *and* $\mathbf{A} = \sum_{i=1}^{s} \lambda_i \mathbf{A}_i$.

Any doubly stochastic matrix can be written as a convex combination of permutation matrices.

E-F-S is Elias-Feinstein-Shannon and F-F is Ford-Fulkerson.

Theorem 7 (Max Flow-Min Cut: E-F-S, 1956 and F-F, 1956) *A network is a directed graph with a **source*** s *and a **target*** t *with each edge assigned an integer called its capacity.*

*An **edge cut*** $[S, S']$ *is the set of edges directed from* S *to* S'*. The **value of an edge cut** is the sum of the capacities.*

*A **flow** is a function* f *on the arcs in which* $f(u,v)$ *is at most the capacity of* (u,v) *and we define* $f^+(v) = \sum_u f(v,u)$ *and* $f^-(v) = \sum_u f(u,v)$ *with the condition that* $f^+(v) = f^-(v)$ *for all* $v \notin \{s,t\}$*. The **value of a flow** is* $f^+(s) - f^-(s)$.

The maximum value of a flow in a network D *is equal to the value of a minimum cut of* D.

Theorem 8 (König's covering theorem, 1933) *An **edge cover** of a graph G is a set of edges such that each vertex that is not isolated is incident to at least one of the edges.*

The size of the largest independent set in a bipartite graph is the size of the smallest edge cover.

Theorem 9 (König-Hall or the Marriage Theorem) *A bipartite graph $G = (A, B; E)$ has a matching that saturates A iff $|N(X)| \geq |X|$ for all $X \subseteq A$.*

Interestingly, I've concluded that König's Theorem is properly referenced with the umlaut, but Dénes Kőnig's name requires the double acute accent. Dénes gave credit to his father Gyula Kőnig for inspiring him to study matchings.

Perhaps these can be called the Fundamental Theorem(s) of Combinatorics. Perhaps not.