Homework 3

Spring 2009 M606:
Enumerative Combinatorics and Partially Ordered Sets

Due Mar. 24, 2009, assigned Mar. 10, 2009

You may ask me any questions in office hours.
LaTeX or other typed solutions are very strongly preferred.

1 Do four (4) of the following:

PROBLEM 1 *In this exercise, we will use generating functions to prove that, for any nonnegative integers n and k that*

$$\binom{n+k+1}{k+1} = \sum_{i=0}^{n} \binom{i+k}{k}. \tag{1}$$

a. Prove that

$$D^k \frac{1}{1-x} \overset{\text{ops}}{\leftrightarrow} \{(n+k)_k\}_0^\infty .$$

b. Prove that

$$f \overset{\text{ops}}{\leftrightarrow} \{a_n\}_0^\infty \qquad \Rightarrow \qquad \frac{f}{1-x} \overset{\text{ops}}{\leftrightarrow} \left\{ \sum_{i=0}^{n} a_i \right\}_0^\infty .$$

c. Use the above to prove that

$$\frac{1}{(1-x)^{k+2}} \overset{\text{ops}}{\leftrightarrow} \left\{ \sum_{i=0}^{n} \binom{i+k}{k} \right\}_0^\infty .$$

d. Observe that $D^{k+1}\frac{1}{1-x} = \frac{(k+1)!}{(x-1)^{k+2}}$ and so

$$\frac{(k+1)!}{(x-1)^{k+2}} \overset{\text{ops}}{\leftrightarrow} \{(n+k+1)_{k+1}\}_0^\infty .$$

e. Combine this information to prove (1).

PROBLEM 2 *Here we provide a good means by which to compute $\sum_{i=0}^{n} i^k$.*

a. *Prove that*

$$\frac{1}{1-x}(xD)^k\frac{1}{1-x} \overset{\text{ops}}{\leftrightarrow} \left\{\sum_{i=0}^{n} i^k\right\}_0^\infty .$$

b. *Prove that*

$$f \overset{\text{ops}}{\leftrightarrow} \{a_n\}_0^\infty \qquad \Rightarrow \qquad x^k f \overset{\text{ops}}{\leftrightarrow} \{a_{n-k}\}_k^\infty .$$

In fact, one can state that it's $\{a_{n-k}\}_0^\infty$*, where* $a_n = 0$ *if* n *is negative.*

c. *Prove that*

$$\frac{1}{1-x}(xD)^2\frac{1}{1-x} = \frac{x+x^2}{(1-x)^4},$$

and use this to prove that

$$\frac{1}{1-x}(xD)^2\frac{1}{1-x} \overset{\text{ops}}{\leftrightarrow} \left\{\binom{(n-1)+4-1}{4-1} + \binom{(n-2)+4-1}{4-1}\right\}_0^\infty .$$

d. *Use this idea to compute* $\sum_{i=0}^{n} i^4$ *in terms of a linear combination of binomial coefficients. It should simplify to*

$$\sum_{i=0}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

PROBLEM 3 *Solve the following difference equations:*

a. $\begin{cases} a_n = 2a_{n-1} - a_{n-2}, & \text{if } n \geq 2; \\ a_0 = 0, a_1 = 1. \end{cases}$

b. $\begin{cases} a_n = -a_{n-2}, & \text{if } n \geq 2; \\ a_0 = 1, a_1 = 0. \end{cases}$

c. $\begin{cases} a_n = -a_{n-1}, & \text{if } n \geq 1; \\ a_0 = 1. \end{cases}$ $\qquad \begin{cases} a_n = -a_{n-1}, & \text{if } n \geq 1; \\ a_0 = 0. \end{cases}$

d. $\begin{cases} a_n = \alpha a_{n-1} + \beta, & \text{if } n \geq 1; \\ a_0, \alpha, \beta & \text{fixed.} \end{cases}$

Hint: *Careful on the last one. There's a different answer for* $\alpha \neq 1$ *and* $\alpha = 1$.

PROBLEM 4

a. *Use the exponential generating function of the sequence* $\{1\}_{n\geq 0}$ *and Cauchy's inequality, as given in class, to show that* $n! \geq (n/e)^n$

b. *Use Cauchy's inequality to prove that*

$$\binom{n}{k} \leq \frac{n^n}{(n-k)^{n-k}k^k}.$$

In both cases, Cauchy's inequality is not enough. One has to choose the value r cleverly; i.e., optimally.

PROBLEM 5 *Let $B(n)$ denote the n^{th} Bell number. That is, if $n \geq 1$, then $B(n)$ denotes the number of partitions of $[n]$ and $B(0) = 1$.*

a. *Use Cauchy's inequality to prove that*

$$\frac{\ln B(n)}{n} \leq \frac{1}{n}\ln(n!) + \frac{e^r - 1}{n} - \ln r$$

for all positive numbers r and any nonnegative integer n.

b. *In 1958, de Bruijn proved that*

$$\frac{\ln B(n)}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2}\right).$$

Prove a weaker estimate by substituting $e^r = \frac{n}{\ln n}$ in part a. to establish that

$$\frac{\ln B(n)}{n} \leq \ln n - \ln\ln n - 1 + o(1).$$

c. *Asymptotically, which is larger? $B(n)$ or $n!$?*

Note: The Cauchy inequality does not give good bounds for the Bell numbers for small values of n, but it looks very good asymptotically. To see this, we can use the following code in Maple:

```
with(combinat,bell):
for n from 100 to 1000 by 100 do
   p:=fsolve(r*exp(r)=n,r);
   print(n.`=n\t\t`, evalf(bell(n)).`=Bell(n)\t\t`,
   n!*exp(exp(p)-1)/p^n.`=Upper bound`); od:
```

It gives $B(1000) \approx 0.29 \times 10^{1928}$ as opposed to an upper bound of 0.59×10^{1930}.