Homework 2

Spring 2009 M606:
Enumerative Combinatorics and Partially Ordered Sets

Due Feb 17, 2009, assigned Feb. 10, 2009

You may ask me any questions in office hours. On Thursday, I can give hints at the beginning of class.

LaTeX or other typed solutions are very strongly preferred.

1 Do four (4) of the following:

PROBLEM 1 *Let $S(n,k)$ denote the Stirling number of the second kind. That is, the number of partitions of $[n]$ into exactly k nonempty parts. Let $B(n)$ denote the Bell numbers. That is, the total number of partitions of $[n]$. Prove the following:*

1. $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$
2. $\sum_{n\geq k} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$
3. $B(n+1) = \sum_{i=0}^{n}\binom{n}{i}B(i)$
4. $\sum_{n\geq 0} B(n)\frac{x^n}{n!} = e^{e^x-1}$

PROBLEM 2 *Prove that, for any sets $A_1, \ldots, A_k$,*

$$\sum_{i=1}^{k}|A_i| \geq \left|\bigcup_{i=1}^{k} A_i\right| \geq \sum_{i=1}^{k}|A_i| - \sum_{i=1}^{k}\sum_{j=i+1}^{k}|A_i \cap A_j|.$$

PROBLEM 3 *Let D_n be the number of derangements of $[n]$. ($D_0 = 1$.) That is, the number of permutations with no fixed point. Prove the following:*

1. *For $n \geq 1$, D_n is the closest integer to $n!/e$.*

2. *For* $n \geq 2$, $D_n = (n-1)\left(D_{n-1} + D_{n-2}\right)$.

3. *For* $n \geq 1$, $D_n = nD_{n-1} + (-1)^n$.

PROBLEM 4 *Prove the following generalization of inclusion-exclusion.*

Theorem 1 *Let* S *be an* n*-element set. Let* V *be the* 2^n*-dimensional vector space (over a field* $\mathbb{F}$*) of all functions* $f : 2^S \to \mathbb{F}$*. Let* $\phi : V \to V$ *be the linear transformation defined by*

$$(\phi f)(T) = \sum_{I \supseteq T} f(I), \qquad \forall T \subseteq S.$$

Then, ϕ^{-1} *exists and is given by*

$$(\phi^{-1} f)(T) = \sum_{I \supseteq T} (-1)^{|I-T|} f(I), \qquad \forall T \subseteq S.$$

PROBLEM 5 *Prove that Theorem 1 can be used to prove that for any finite set* Ω *and subsets* $A_1, \ldots, A_n$*,*

$$\left|\Omega - \bigcup_{i=1}^{n} A_i\right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left|A_I\right|,$$

where $A_I = \bigcap_{i \in I} A_i$.

PROBLEM 6 *Let* n *be a positive integer and let* $f(n)$ *denote the number of subsets of* $\mathbb{Z}/n\mathbb{Z}$ *whose elements sum to* 0 *in* $\mathbb{Z}/n\mathbb{Z}$*. For example,* $f(4) = 4$ *because it corresponds to the subsets:* $\emptyset$, $\{0\}$, $\{1,3\}$ *and* $\{0,1,3\}$*. Show that*

$$f(n) = \frac{1}{n} \sum_{\substack{d \mid n \\ d \text{ odd}}} \phi(d) 2^{n/d}.$$

The Euler phi function, $\phi(d)$*, is the number of positive integers at most* d *that are relatively prime to* d*.*