Homework 1

Spring 2009 M606:
Enumerative Combinatorics and Partially Ordered Sets

Due Feb 03, 2009, assigned Jan. 27, 2009

You may ask me any questions in office hours. On Thursday, I can give hints at the beginning of class.

LaTeX or other typed solutions are very strongly preferred.

1 Do at least three (3) of the following:

PROBLEM 1 *Prove the multinomial theorem. That is:*

Theorem 1 *Let $n \in \mathbb{N}$ and r be a positive integer.*

$$(x_1 + \cdots + x_r)^n = \sum_{\substack{k_1 + \cdots + k_r = n \\ k_i \geq 0, \forall i \in [r]}} \binom{n}{k_1, \ldots, k_r} x_1^{k_1} \cdots x_r^{k_r},$$

where the multinomial coefficient $\binom{n}{k_1,\ldots,k_r}$ satisfies $\binom{n}{k_1,\ldots,k_r} = \frac{n!}{k_1! \cdots k_r!}$.

PROBLEM 2 *Let $1 \leq k \leq n$. Prove*

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k.$$

PROBLEM 3 *Show that $\ln n! \sim n \ln n$ but that $n! \not\sim n^n$.*

PROBLEM 4 *Let m, n be nonnegative integers. Give a **combinatorial proof** of the fact that*

$$\left(\!\binom{n}{m}\!\right) = \left(\!\binom{m+1}{n-1}\!\right).$$

2 Do at least two (2) of the following:

PROBLEM 5 *Prove that if $j \geq 2$, then*

$$\ln j \geq \int_{j-1/2}^{j+1/2} \ln x \, dx$$

and use it to prove that

$$n! \geq \sqrt{\frac{8}{27}e^3}\sqrt{n}\left(\frac{n}{e}\right)^n.$$

PROBLEM 6

a. *Use the generalized binomial theorem to prove that*

$$(1-4x)^{-1/2} = \sum_{n\geq 0}\binom{2n}{n}x^n.$$

b. *Compute* $\sum_{n\geq 0}\binom{2n-1}{n}x^n$.

c. *Use (a) to prove that, for every positive integer* n*,*

$$\sum_{i=0}^{n}\binom{2i}{i}\binom{2(n-i)}{n-i} = 4^n.$$

PROBLEM 7 *Let* $p_k(n)$ *denote the number of partitions of integer* n *into* k *parts. Fix* $t \geq 0$*. Show that as* $n \to \infty$*,* $p_{n-t}(n)$ *becomes eventually constant. What is the constant* $f(t)$*? What is the least value of* n *for which* $p_{n-t}(n) = f(t)$*? Your proof should be combinatorial.*

PROBLEM 8 *Let* n *be a positive integer and let* $f(n)$ *denote the number of subsets of* $\mathbb{Z}/n\mathbb{Z}$ *whose elements sum to* 0 *in* $\mathbb{Z}/n\mathbb{Z}$*. For example,* $f(4) = 4$ *because it corresponds to the subsets:* $\emptyset$, $\{0\}$, $\{1,3\}$ *and* $\{0,1,3\}$*. Show that*

$$f(n) = \frac{1}{n}\sum_{\substack{d\,\mid\, n\\ d \text{ odd}}}\phi(d)2^{n/d}.$$

The Euler phi function, $\phi(d)$*, is the number of positive integers at most* d *that are relatively prime to* d*.*