Take-home Exam 2

Spring 2009 M606:
Enumerative Combinatorics and Partially Ordered Sets

Due May 7, 2009, assigned May 5, 2009

Please hand in solutions for all problems. You must work entirely on your own, but you may use class notes. You may also use the notes I post on the web or the textbooks mentioned in the syllabus. Use no other sources.

The exam is due by **3pm on Thursday, May 7**. I will be in my office 428 Carver from 2pm to 3pm to collect them and I will be around Carver much of the day.

LaTeX or other typed solutions are preferred, but neat writing is sufficient.

PROBLEM 1 (10 points) *Compute the rank generating function for each of the following posets:*

a. Π_n*, the poset of partitions ordered by refinement.*

b. The poset of permutations of $[n]$ *under the weak order.*

Hint: *Dig through old notes. Both of these functions were computed before.*

PROBLEM 2 (10 points) *Let* $\mathbf{2}^{[n]}$ *represent the Boolean lattice of dimension* n*.*

a. Compute the Möbius function $\mu(S,T)$*.*

b. Use the Dual Möbius Inversion Formula to prove inclusion-exclusion.

PROBLEM 3 (10 points) *Compute the following members of* $\binom{\mathbb{N}}{4}$ *in colexicographical order:*

- *The* 999^{th} *element.*
- *The* 1000^{th} *element.*
- *The* 1001^{st} *element.*
- *The* 1002^{nd} *element.*
- *The* 1003^{rd} *element.*

PROBLEM 4 (10 points)

- *Let* $\mathcal{A} \subseteq 2^{[n]}$ *be a maximal intersecting family. An intersecting family has the property that for any distinct* $A_1, A_2 \in \mathcal{A}$*, then* $A_1 \cap A_2 \neq \emptyset$*. Prove that* $\mathcal{A}$ *is monotone increasing. What is* $|\mathcal{A}|$*, given that it is maximal?*

- *Let* $\mathcal{C} \subseteq 2^{[n]}$ *be a maximal family with the property that for any distinct* $C_1, C_2 \in \mathcal{C}$*, then* $C_1 \cup C_2 \neq [n]$*. Prove that* $\mathcal{C}$ *is monotone decreasing. What is* $|\mathcal{C}|$*, given that it is maximal?*

- *Let* $\mathcal{F} \subseteq 2^{[n]}$ *have the property that for* $F_1, F_2 \in \mathcal{F}$*,* $F_1 \cap F_2 \neq \emptyset$ *and* $F_1 \cup F_2 \neq [n]$*. Prove that* $|\mathcal{F}| \leq 2^{n-2}$*.*

DO ONE OF THE FOLLOWING:

PROBLEM 5 (10 points) *A lattice is called modular if, for every* $a \leq c$*, the following holds:*

$$a \vee (b \wedge c) = (a \vee b) \wedge c. \tag{1}$$

- *Prove that in any lattice – modular or not – if* $a \leq c$*, then*

$$a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

- *Prove that* N_5 *(shaped like a pentagon) is not a modular lattice but that* M_5 *(also known as* Π_3*) is a modular lattice.*

- *Prove that every distributive lattice is modular.*

- *Prove that if a lattice is not modular, then it contains a sublattice isomorphic to* N_5*.*

PROBLEM 6 (10 points)

- *Prove that if* $r \geq 3$*,* $\mathcal{A} \subset \binom{\mathbb{N}}{r}$*,* $|\mathcal{A}| = \binom{m}{r}$ *where* $r \leq m$*, and* $|\partial\mathcal{A}| \leq \binom{m}{r-1}$ *then* $\mathcal{A} \cong \binom{[m]}{r}$*.*

- *Prove that if* $r \geq 3$*,* $\mathcal{A} \subset \binom{\mathbb{N}}{r}$*,* $|\mathcal{A}| = b^{(r)}(m_r, m_{r-1})$ *where* $r \leq m_{r-1} < m_r$*, and* $|\partial\mathcal{A}| \leq b^{(r-1)}(m_r, m_{r-1})$ *then* $\mathcal{A} \cong \mathcal{B}^{(r)}(m_r, m_{r-1})$*.*