21-801 Extremal Graph Theory	**Lecturer: Ryan Martin**
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A theorem from last class states that if a graph G_n on n nodes is the union of n induced matchings, than $e(G_n) = o(n^2)$. Avi Wigderson and Johan Håstad gave the construction of a family of graphs G_n that are the union of n induced matchings for which $e(G_n) \geq n^{2-o(1)}$. The following simple construction provides instead an example for which the number of edges is at least $n\sqrt{n}$: consider the projective plane as a hypergraph and put a matching in each hyperedge.

Theorem 1 (Ramsey-Turán theorem for K_4, Szemerédi,1972) *If G_n contains no K_4 and only $o(n)$ independent vertices, then $e(G_n) < \frac{1}{8}n^2 + o(n^2)$ (i.e. $\alpha(G_n) = o(n)$)*

If we ignore the independence number of G_n, Turán's Theorem would give us $e(G_n) < (1-\frac{1}{3})\frac{n^2}{3} = \frac{1}{3}n^2$. The above result is tight, as shown by Bollobás and Erdős, who gave a construction achieving $e(G_n) > \frac{1}{8}n^2$.

Theorem 2 (Bollobás, Erdős, Simonovits, Szemerédi,1978) *Let t be an arbitrary natural number and c an arbitrary positive real number. Then there exists a positive integer n_0 such that, if $n \geq n_0$ and G_n is a graph on n nodes, then either G_n can be turned into a bipartite graph by deleting cn^2 edges or there exists a blow-up odd cycle $C_{2k+1}(t) \subseteq G_n$ for some k satisfying $2k+1 < 1/c$.*

(The number of edges in $C_{2k+1}(t)$ is $(2k+1)t^2 < t^2/c$.)

Definition 1 *A graph $G = (V, E)$ has the property (γ, δ, σ) if for every $S \subseteq V$ with $|S| > \gamma|V|$ the induced graph $G(S)$ satisfies*

$$(\sigma - \delta)\binom{|S|}{2} \leq e(G(S)) \leq (\sigma + \delta)\binom{|S|}{2}.$$

Theorem 3 (Rödl, 1986) *For every positive integer k and every $\sigma > 0$, $\delta > 0$ such that $\delta < \sigma < 1 - \delta$, there exists γ and a positive integer n_0 such that every graph $G - n$ with $n \geq n_0$ vertices satisfying the property (γ, δ, σ) contains all graphs with k vertices as induced subgraphs.*

Corollary 1 *For every graph L there exists a graph H such that, for any 2-coloring of the edges of H, H contains an induced monochromatic subgraph isomorphic to L.*

The corollary was actually proved earlier ('76), but Rödl's Theorem yields an easy proof.

Rödl also proved the following.

Theorem 4 *For every positive integer k, for every σ and γ, there exists $\delta > 0$ and a positive integer n_0 such that every graph G_n on $n \geq n_0$ vertices having the property (γ, δ, σ) contains all graphs with k nodes as induced subgraphs.*

Definition 2 *$G = (V, E)$ is a 2-diameter critical graph if* $\mathrm{diam}(G) = 2$ *and* $\mathrm{diam}(G \setminus e) > 2$ *for every $e \in E$.*

Conjecture 1 (Murty and Simon, 1974) *If G_n is a 2-diameter critical graph then*

$$e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

with equality if and only if $G \simeq K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$.

The conjecture is still open, but Füreti, in 1992, was able to show, via the Regularity Lemma, that there exists a positive integer n_0 such that the conjecture is true for every graph G with at least n_0 nodes.

Theorem 5 *For every $\Delta, \beta > 0$ there exists $c > 0$ and a positive integer n_0 such that for every graph G_n on $n \geq n_0$ nodes, if $e(G_n) > \beta n$ then G_n contains as subgraph all bipartite graphs H with $|V(H)| \leq cn$ and $\Delta(G) \leq \Delta$.*

Proof: (sketch) Apply the Regularity Lemma to get an ϵ-regular pair with sufficient density. Apply the Key Lemma, where the graph R is simply two nodes joined by an edge, and $t = cn$. Since H can be embedded into $R(t)$, then it is also embeddable in the ϵ regular pair (one has to be careful with the choice of the constants). ■

Theorem 6 (Chvátal. Rödl, Szemerédi, 1983) *For every $\Delta > 0$, there exists $c > 0$ and a positive integer n_0 such that, for every graph G_n with $n \geq n_0$ nodes and every graph H with $|V(H)| \leq cn$ and $\Delta(H) \leq \Delta$, then $H \subseteq G$ or $H \subseteq \overline{G}$.*

Proof: Let $r = \chi(H) \quad (\leq \Delta(H) + 1)$. Let ϵ be small, apply the Regularity Lemma, delete the edges between all irregular pairs. Consider G_r, the reduced graph of G. Color the super-edges of G_r RED if the density of the corresponding pair is at least 1/2, and BLUE otherwise. Observe that the complement of an ϵ-regular pair is ϵ-regular. The Theorem will follow from the following.

Fact: *For all r, there exists ϵ and k_0 such that, if we bicolor the edges of a graph G with $k \geq k_0$ nodes and at least $(1-\epsilon)\binom{k}{2}$ edges, then there exists a monochromatic r-clique.*

Since there are at most $\epsilon\binom{k}{2}$ ϵ-irregular pairs, then G_r has a monochromatic r-clique. The result follows by applying the Key Lemma to the subgraph of G represented by the r-clique or to its complement (depending on whether the r-clique is, respectively, RED or BLUE). ■

Conjecture 2 (Erdős, Sós,1963) *Every graph on* n *vertices with at least* $\frac{k-1}{2}n$ *edges contains as subgraphs all trees with* k *edges.*

The following Theorem is related to the previous conjecture.

Theorem 7 (Erdős-Sós, approximate form, by Ajtai, Komlós and Szemerédi, 1991) *For every* $\epsilon > 0$ *there exists a positive integer* k_0 *such that, for every* $k \geq k_0$*, every graph with average degree at least* $(1+\epsilon)k$ *contains, as subgraphs, all trees with* k *edges.*

The proof is divided into two parts:

Dense case: uses the Regularity Lemma;

Sparse case: uses a "sparse version" of the Regularity Lemma that does not seem to be easily applicable elsewhere.

Another conjecture with the along the same line is the following.

Conjecture 3 (Loebl) *If* G *is an* n *vertex graph, such that at least* $\frac{n}{2}$ *vertices have degree at least* $\frac{n}{2}$*, then* G *contains, as subgraphs, all trees with at most* $\frac{n}{2}$ *edges.*