21-484 Graph Theory	Homework 5 Solution
	Due: April 17, 2002

Problem 1. Show that if $G \cap H$ is complete then $\pi_k(G \cup H)\pi_k(G \cap H) = \pi_k(G)\pi_k(H)$.

Proof. This is because for any fixed coloring C_1 of H, it can be mapped to a coloring of H that colors $G \cap H$ with a coloring C_2. Because $G \cap H$ is complete, all colorings of $G \cap H$ will have distinct colors. Thus, every coloring C_1 of H will be mapped to exactly one coloring of H that has $G \cap H$ colored with coloring C_2. So, given a fixed coloring of $G \cap H$, the number of possible colorings of H is the same quantity, call it $\Gamma_k(H \mid G \cap H)$. Thus, $\pi_k(H) = \pi_k(G \cap H)\Gamma_k(H \mid G \cap H)$. So, $\Gamma_k(H \mid G \cap H) = \frac{\pi_k(H)}{\pi_k(G \cap H)}$. Therefore, to color $G \cup H$, just color G. Then, given that coloring of G, $G \cap H$ will have fixed colors. Hence, there are $\frac{\pi_k(H)}{\pi_k(G \cap H)}$ ways to color what remains of H. Thus, $\pi_k(G \cup H) = \pi_k(G) \cdot \frac{\pi_k(H)}{\pi_k(G \cap H)}$.

Alternatively, we obverse that to color $G \cup H$, color G and H. Every vertex in $G \cap H$ will receive a "G-color" and an "H-color". But, since $G \cap H$ is complete, only $\frac{\pi_k(G \cap H)}{(\pi_k(G \cup H))^2}$ of these will be proper colorings of $G \cup H$. Thus, we have that:

$$\pi_k(G \cup H) = \frac{\pi_k(G)\pi_k(H)}{\pi_k(G \cap H)}$$

□

Problem 2. Let a graph G be given. Assume G has a cut-edge e where $G - e$ has components H_1 and H_2. Compute $\pi_k(G)$ in terms of k, $\pi_k(H_1)$ and $\pi_k(H_2)$.

We know that $\pi_k(G) = \pi_k(G - e) - \pi_k(G \cdot e) = \pi_k(H_1)\pi_k(H_2) - \pi_k(H')$. H' is $H_1 \cup H_2$ where $|H_1 \cap H_2| = 1$. Therefore,

$$\pi_k(G) = \pi_k(H_1)\pi_k(H_2) - \frac{\pi_k(H_1)\pi_k(H_2)}{\pi_k(H_1 \cap H_2)} = \left(1 - \tfrac{1}{k}\right)\pi_k(H_1)\pi_k(H_2)$$

Problem 3. Show that if G is a nonempty regular simple graph with ν odd, then $\chi' = \Delta + 1$.

Proof. Let G be a non-empty and Δ-regular graph. If we try to Δ-color the edges, each color class is a matching. Also if G were Δ-edge-colorable, every vertex would be adjacent to a member of each color class. Since every color class is a matching, and ν is odd, each color class must be nonadjacent to some vertex, a contradiction. □

Problem 4. Show that for all k, ℓ, $r(k, \ell) = r(\ell, k)$.

Proof. There is a graph G on $r(k, \ell) - 1$ vertices with no k-cycle or ℓ-independent set. Then, G^c has no ℓ-cycle or k-independent set, which implies that $r(\ell, k) \geq r(k, \ell)$. Similarly, if there were a graph on $r(k, \ell)$ vertices that has no ℓ-clique or k-independent set, its complement would have no k-clique or ℓ-independent set, which is not true by definition. Thus, $r(\ell, k) \geq r(k, \ell)$. So, $r(\ell, k) = r(k, \ell)$ for all k, ℓ. □

Problem 5. The composition of simple graphs G and H is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') iff either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. (a) Show that $\alpha(G[H]) \leq \alpha(G)\alpha(H)$. (b) Show that $r(mn + 1, mn + 1) - 1 \geq (r(m + 1, m + 1) - 1)(r(n + 1, n + 1) - 1)$. (c) Deduce that $r(2^n + 1, 2^n + 1) \geq 5^n + 1$ for all $n \geq 0$.

Proof of Part (a). Let I be an independent set of $G[H]$. Since $(u,v) \sim (u',v')$ iff either $u \sim u'$ or $u = u'$ and $v \sim v'$, then $(u,v) \nsim (u',v')$ iff both $u \nsim u'$ and either $u \neq u'$ or $v \nsim v'$. Let S be the set of all s such that $\exists v$ with $(s,v) \in I$. S must be an independent set in G because if $s_1, s_2 \in S$ and $s_1 \neq s_2$, $s_1 \nsim s_2$. Given $s \in S$, let T_s be the set of all t such that $(s,t) \in I$. T_s must be an independent set, because if $(s,t_1),(s,t_2) \in I$ with the set $t_1 \sim t_2$. So,

$$|I| = \sum_{s \in S} |T_s| \leq \sum_{s \in S} \alpha(H) = \alpha(H)\,|S| \leq \alpha(H)\alpha(G)$$

□

Proof of Part (b). Let G^c denote the complement of a graph G.

Claim. $G^c\big[H^c\big] = \big(G[H]\big)^c$.

Proof of Claim. Let $(u,v),(u',v') \in V\Big(\big(G[H]\big)^c\Big)$ and distinct. If $(u,v) \sim (u',v')$ in $\big(G[H]\big)^c$, then both $uu' \notin E(G)$ and either $u \neq u'$ or $vv' \notin E(H)$. Since G is simple, this means either $uu' \in E\big(G^c\big)$ or $u = u'$ and $vv' \in E\big(H^c\big)$.

Thus, $(u,v) \sim (u',v')$ in $G^c\big[H^c\big]$.

A similar argument establishes that $(u,v) \sim (u',v')$ in $G^c\big[H^c\big]$ implies that $(u,v) \sim (u',v')$ in $\big(G[H]\big)^c$. □

To prove part (b), observe that $\alpha\big(G^c\big)$ is the size of the largest clique in a graph G. Let G be a $r(m+1,m+1)$ Ramsey graph and let H be a $r(n+1,n+1)$ Ramsey graph.

By definition, G has $r(m+1,m+1)-1$ vertices, $\alpha(G) \leq m$ and $\alpha\big(G^c\big) \leq m$. Also, H has $r(n+1,n+1)-1$ vertices, $\alpha(H) \leq n$ and $\alpha\big(H^c\big) \leq n$.

$$\alpha\big(G[H]\big) \leq \alpha(G)\alpha(H) \leq mn$$

Using the claim,

$$\alpha\Big(\big(G[H]\big)^c\Big) \leq \alpha\big(G^c\big[H^c\big]\big) \leq \alpha\big(G^c\big)\alpha\big(H^c\big) \leq mn$$

Thus, the graph $G[H]$ is a graph on

$$\big(r(m+1,m+1)-1\big) \times \big(r(n+1,n+1)-1\big)$$

vertices. By definition, this quantity must be at most

$$r(mn+1,mn+1)-1$$

□

Proof of Part (c). Proceed by induction on n. We know that $r(2^0+1,2^0+1) \geq 5^0+1$, and $r(2^1+1,2^1+1) = r(3,3) = 6 \geq 5^1+1 = 6$. Let $n \geq 2$, and suppose that $r(2^{n-1}+1,2^{n-1}+1) \geq 5^{n-1}+1$. By part (b), let $k := 2$, and $\ell = 2^{n-1}$, we have:

$$\begin{aligned} r(2 \cdot 2^{n-1}+1, 2 \cdot 2^{n-1}+1) - 1 &\geq [r(2+1,2+1)-1][r(2^{n-1}+1,2^{n-1}+1)-1] \\ &\geq \big(r(3,3)-1\big)5^{n-1} = 5 \cdot 5^{n-1} = 5^n \end{aligned}$$

So, $r(2^n+1,2^n+1) \geq 5^n+1$. □