Hopf Algebras of Dimension p^2

Siu-Hung Ng
Mathematics Department, Towson University, Baltimore, MD 21252

Abstract

Let p be a prime number. It is known that any non-semisimple Hopf algebra of dimension p^2 over an algebraically closed field of characteristic 0 is isomorphic to a Taft algebra. In this exposition, we will give a more direct alternative proof to this result.

0 Introduction

Let p be an odd prime and k an algebraically closed field of characteristic 0. It was shown independently in [AS98b], [BDG99] and [Gel98] that there exist infinitely many isomorphism classes of Hopf algebras of dimension p^4 over k. These results give counterexamples to a Kaplansky conjecture that there were only finitely many types in each dimension [Kap75]. However, for some special dimensions, there are, indeed, only finitely many isomorphism classes of Hopf algebras. For example, Hopf algebras of dimension p over k are the group algebra $k[\mathbf{Z}_p]$ and hence there is only one type of Hopf algebras of dimension p [Zhu94].

If H is a semisimple Hopf algebra of dimension p^2 over k, then H is isomorphic to $k[\mathbf{Z}_{p^2}]$ or $k[\mathbf{Z}_p \times \mathbf{Z}_p]$ by [Mas96]. A more general result for semisimple Hopf algebras of dimension pq, where p, q are odd prime, is obtained by [EG98]. However, there are very few examples of non-semisimple Hopf algebras of these dimensions. In fact, there are no non-semisimple Hopf algebras over k of dimension 15, 21 or 35 (cf. [AN01]).

For non-semisimple Hopf algebras of dimension p^2, the only known example during the last three decades is the Taft algebras (cf. [Mon98, Section 5] and [Taf71]). Let $\omega \in k$ be a primitive nth root of unity. The Taft algebra associated with ω, denoted by $T(\omega)$, is the Hopf algebra generated by x and a, as a k-algebra, subject to the relations

$$a^n = 1, \quad ax = \omega xa, \quad x^n = 0\,,$$

with the coalgebra structure, and the antipode of $T(\omega)$ given by

$$\Delta(a) = a \otimes a, \quad S(a) = a^{-1}, \quad \varepsilon(a) = 1,$$
$$\Delta(x) = x \otimes a + 1 \otimes x, \quad S(x) = -xa^{-1}, \quad \varepsilon(x) = 0\,.$$

The Hopf algebra $T(\omega)$ is not semisimple and $\dim T(\omega) = n^2$. It is also known that $T(\omega) \cong T(\omega)^*$ as Hopf algebras, and that $T(\omega) \cong T(\omega')$ only if $\omega = \omega'$.

The question whether the Taft algebras are the only non-semisimple Hopf algebras of dimension p^2 remained open for the last three decades, and it was repeatedly asked by Susan Montgomery in several conferences. However, there were some partial answers to the question. Andruskiewitsch and Chin proved that non-semisimple *pointed* Hopf algebras of dimension p^2 are indeed Taft algebras (cf. [Şte97]). Using this result, Andruskiewitsch and Schneider proved in [AS98a] that if H is non-semisimple Hopf algebra with antipode S of dimension p^2 such that the order of S^2 is p, then H is isomorphic to a Taft algebra. In the course of studying Hopf algebras of dimension pq where p, q are odd primes, the author proved in [Ng02], using the later results, that non-semisimple Hopf algebras of dimension p^2 are indeed Taft algebras. Hence, the classification of Hopf algebras of dimension p^2 is completed and they are the following list of $p + 1$ non-isomorphic Hopf algebras over k of dimension p^2

(a) $k[\mathbf{Z}_{p^2}]$;

(b) $k[\mathbf{Z}_p \times \mathbf{Z}_p]$;

(c) $T(\omega)$, $\omega \in k$ a primitive pth of unity.

In this paper, we will give a more direct proof to the result that if H is a non-semisimple Hopf algebra with antipode S of dimension p^2 over k, then $o(S^2) = p$ and hence by [AS98a], H is isomorphic to a Taft algebra. The statement for the case when $p = 2$ is proven in [Kap75].

1 Notation and Preliminaries

Throughout this paper, p is an odd prime, k is an algebraically closed field of characteristic 0 and H is a finite-dimensional Hopf algebra over k with antipode S. Its comultiplication and counit are, respectively, denoted by Δ and ε. A non-zero element $a \in H$ is called group-like if $\Delta(a) = a \otimes a$. The set of all group-like elements $G(H)$ of H is a linearly independent set, and it forms a group under the multiplication of H. For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to the references [Swe69] and [Mon93].

The order of the antipode is of fundamental importance to the semisimplicity of H. We recall some important results on the antipodes of finite-dimensional Hopf algebras.

Theorem 1.1 [LR87, LR88] *Let H be a finite-dimensional Hopf algebra with antipode S over a field of characteristic 0. Then the following statements are equivalent:*

(i) *H is semisimple.*

(ii) *H^* is semisimple.*

(iii) $\mathrm{Tr}(S^2) \neq 0$.

(iv) $S^2 = id_H$.

Let $\lambda \in H^*$ be a non-zero right integral of H^* and let $\Lambda \in H$ be a non-zero left integral of H. There exists $\alpha \in \mathrm{Alg}(H, k) = G(H^*)$, independent of the choice of Λ, such that $\Lambda a = \alpha(a)\Lambda$ for $a \in H$. Likewise, there is a group-like element $g \in H$, independent of the choice of λ, such that $\beta\lambda = \beta(g)\lambda$ for $\beta \in H^*$. We call g the distinguished group-like element of H and α the distinguished group-like element of H^*. Then we have a formula for S^4 in terms of α and g [Rad76]:

$$S^4(a) = g(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1})g^{-1} \quad \text{for } a \in H \,, \tag{1.1}$$

where $\rightharpoonup$ and $\leftharpoonup$ denote the natural actions of the Hopf algebra H^* on H described by

$$\beta \rightharpoonup a = \sum a_{(1)}\beta(a_{(2)}) \quad \text{and} \quad a \leftharpoonup \beta = \sum \beta(a_{(1)})a_{(2)}$$

for $\beta \in H^*$ and $a \in H$. Hence, we have

$$o(S^4) \mid \mathrm{lcm}(o(g), o(\alpha)) \,. \tag{1.2}$$

The Hopf algebra H (respectively H^*) is said be *unimodular* if α (resp. g) is trivial. If both H and H^* are unimodular, then $S^4 = id_H$.

Lemma 1.2 [AS98a] [LR95] *Let H be a finite-dimensional non-semisimple Hopf algebra with antipode S of odd dimension over k. Then, H and H^* can not be both unimodular and $S^4 \neq id_H$.*

Proof. It follows immediately from [LR95, Theorem 2.1] or [AS98a, Lemma 2.5] . ■

Following [Ng02], we define the index of H to be the least positive integer n such that

$$g^n = 1 \quad \text{and} \quad S^{4n} = id_H \,.$$

Lemma 1.3 *Let H be a finite-dimensional non-semisimple Hopf algebra with antipode S of dimension p^2 where p is an odd prime. Suppose that g and α are the distinguished group-like elements of H and H^* respectively. Then*

$$o(S^4) = \operatorname{lcm}(o(g), o(\alpha)) = p\,.$$

In particular, H is of index p.

Proof. Since H is not semisimple and $\dim H$ is odd, by Lemma 1.2,

$$S^4 \neq id_H \tag{1.3}$$

and H, H^* cannot be both unimodular. Hence $\operatorname{lcm}(o(g), o(\alpha)) > 1$. By Nichols-Zoeller Theorem [NZ89],

$$o(\alpha) \,|\, p^2 \quad \text{and} \quad o(g) \,|\, p^2 \,.$$

If $o(g) = p^2$ or $o(\alpha) = p^2$, then

$$H \cong k[\mathbf{Z}_{p^2}] \quad \text{or} \quad H \cong k[\mathbf{Z}_{p^2}]^*$$

and hence H is semisimple. Therefore,

$$o(g) \,|\, p \quad \text{and} \quad o(\alpha) \,|\, p$$

and hence $\operatorname{lcm}(o(g), o(\alpha)) = p$. Since $o(S^4) \neq 1$ and $o(S^4) \,|\, \operatorname{lcm}(o(g), o(\alpha)) = p$, we have $o(S^4) = p$. ■

2 Eigenspace Decompositions for Hopf algebras of Odd Index

Let H be a finite-dimensional Hopf algebra with antipode S of odd index $n > 1$. Suppose that g and α are the distinguished group-like elements of H and H^* respectively. The element g defines a coalgebra automorphism $r(g)$ on H as follows:

$$r(g)(a) = ag \quad \text{for} \quad a \in H\,.$$

Since S^2 is an algebra automorphism on H and $S^2(g) = g$, we have

$$S^2 \circ r(g) = r(g) \circ S^2$$

and so $r(g)$ and S^2 are simultaneously diagonalizable. Let $\omega \in k$ be a primitive nth root of unity. The eigenvalues of S^2 are of the form $(-1)^a\omega^i$ and the eigenvalues of $r(g)$ are of the form ω^j. Define

$$H^\omega_{a,i,j} = \{u \in H \mid S^2(u) = (-1)^a\omega^i u\,, ug = \omega^j u\} \text{ for any } (a,i,j) \in \mathbf{Z}_2 \times \mathbf{Z}_n \times \mathbf{Z}_n\,.$$

We will simply write $\mathcal{K}_n$ for the group $\mathbf{Z}_2 \times \mathbf{Z}_n \times \mathbf{Z}_n$, write $H^\omega_{(a,i,j)}$ for $H^\omega_{a,i,j}$ and $\mathbf{a}$ for (a,i,j). Since S^2 and $r(g)$ are simultaneously diagonalizable, we have the decomposition

$$H = \bigoplus_{\mathbf{a} \in \mathcal{K}_n} H^\omega_{\mathbf{a}}\,. \tag{2.1}$$

Note that $H^\omega_{\mathbf{a}}$ could be zero.

Since $g^n = 1$ and α is an algebra map, $\alpha(g)$ is a nth root of 1. There exists a unique element $x(\omega, H) \in \mathbf{Z}_n$ such that

$$\omega^{x(\omega,H)} = \alpha(g)\,.$$

Using the eigenspace decomposition of H in (2.1), the diagonalization of a left integral Λ of H admits an interesting form.

Lemma 2.1 [Ng02] *Let H be a finite-dimensional Hopf algebra with antipode S of odd index $n > 1$, $\omega \in k$ primitive nth root of 1, and $x = x(\omega, H)$. Suppose Λ is a non-zero left integral of H. Then we have*

$$\Delta(\Lambda) = \sum_{\mathbf{a} \in \mathcal{K}_n} \left(\sum u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}}\right) \tag{2.2}$$

where $\mathbf{x} = (0, -x, x)$ and $\sum u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}} \in H^\omega_{\mathbf{a}} \otimes H^\omega_{-\mathbf{a}+\mathbf{x}}$. Moreover, if λ is a right integral of H^* such that $\lambda(\Lambda) = 1$, then

$$\dim H^\omega_{\mathbf{a}} = \sum \lambda(S(v_{\mathbf{x}-\mathbf{a}})u_{\mathbf{a}})$$

for any $\mathbf{a} \in \mathcal{K}_n$.

3 Antipodes of Hopf Algebras of Dimension p^2

In [Ng02], the author proved that if H is a non-semisimple Hopf algebra with antipode S of dimension p^2, then $o(S^2) = p$ and hence, by [AS98a], H is isomorphic to a Taft algebra. In this section, we will provide a more direct proof to the result using the following lemmas.

Lemma 3.1 [Ng02] Let H be a finite-dimensional Hopf algebra of index p where p is an odd prime, and let $\omega \in k$ be a primitive pth root of 1. If H^* is not unimodular, then for any $j \in \mathbf{Z}_p$,

(i) there exists an integer d_j such that

$$\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j} = d_j \quad \text{for all } i \in \mathbf{Z}_p\,,$$

(ii) and

$$\sum_{(a,i)\in \mathbf{Z}_2 \times \mathbf{Z}_p} \dim H^{\omega}_{a,i,j} = \frac{\dim H}{p}\,.$$

Lemma 3.2 *Let H be a finite-dimensional Hopf algebra with antipode S of index p where p is an odd prime, and $\omega \in k$ a primitive pth root of 1. If H^* is not unimodular, then*

$$\dim H^{\omega}_{0,-x,j} \neq 0$$

for all $j \in \mathbf{Z}_p$ where $x = x(\omega, H)$.

Proof. Let λ be a right integral of H^* and Λ a left integral of H such that $\lambda(\Lambda) = 1$. By Lemma 2.1,

$$\Delta(\Lambda) = \sum_{\mathbf{a}\in\mathcal{K}_p} \Big(\sum u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}}\Big)$$

where $\mathbf{x} = (0, -x, x)$ and $\sum u_{\mathbf{a}} \otimes v_{-\mathbf{a}+\mathbf{x}} \in H^{\omega}_{\mathbf{a}} \otimes H^{\omega}_{-\mathbf{a}+\mathbf{x}}$. Note that for any $(a, i, j) \in \mathcal{K}_p$,

$$\sum S(v_{a,-x-i,x-j})u_{a,i,j} \in H^{\omega}_{0,-x,j}\,. \tag{3.1}$$

It suffices to show that for any $j \in \mathbf{Z}_p$, there exists $(a, i) \in \mathbf{Z}_2 \times \mathbf{Z}_p$ such that

$$\sum S(v_{a,-x-i,x-j})u_{a,i,j} \neq 0.$$

Suppose not. Then there exists $j \in \mathbf{Z}_p$ such that for all $(a, i) \in \mathbf{Z}_2 \times \mathbf{Z}_p$,

$$\sum S(v_{a,-x-i,x-j})u_{a,i,j} = 0.$$

Then, by Lemma 2.1,

$$\dim H^{\omega}_{a,i,j} = \sum \lambda(S(v_{a,-x-i,x-j})u_{a,i,j}) = 0$$

for all $(a, i) \in \mathbf{Z}_2 \times \mathbf{Z}_p$. Thus,

$$\sum_{(a,i)\in \mathbf{Z}_2 \times \mathbf{Z}_p} \dim H^{\omega}_{a,i,j} = 0$$

but this contradicts Lemma 3.1(ii). ∎

Lemma 3.3 *Let H be a non-semisimple Hopf algebra with antipode S of dimension p^2 where p is a odd prime, and let $\omega \in k$ be a primitive pth root of 1. Then for any $i, j \in \mathbf{Z}_p$,*

$$\dim H^{\omega}_{0,i,j} = 1 \quad \textit{and} \quad \dim H^{\omega}_{1,i,j} = 0\,.$$

Proof. For $j \in \mathbf{Z}_p$, by Lemma 3.1(i),

$$\sum_{i \in \mathbf{Z}_p} \dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j} = p d_j\,.$$

Hence, by Lemma 3.1(ii), $|d_j| = 0$ or 1 for all $j \in \mathbf{Z}_p$. If $d_j = 0$, for any $i \in \mathbf{Z}_p$, we have

$$\dim H^{\omega}_{0,i,j} = \dim H^{\omega}_{1,i,j}$$

and hence

$$p = \sum_{i \in \mathbf{Z}_p} \dim H^{\omega}_{0,i,j} + \dim H^{\omega}_{1,i,j} = 2 \sum_{i \in \mathbf{Z}_p} \dim H^{\omega}_{0,i,j}\,.$$

This contradicts that p is odd. If $d_j = -1$, then

$$\sum_{i \in \mathbf{Z}_p} H^{\omega}_{0,i,j} - H^{\omega}_{1,i,j} = -p\,.$$

By Lemma 3.1(ii),

$$\sum_{i \in \mathbf{Z}_p} \dim H^{\omega}_{0,i,j} = 0 \quad \text{and} \quad \sum_{i \in \mathbf{Z}_p} \dim H^{\omega}_{1,i,j} = p$$

but the first equation contradicts Lemma 3.2. Therefore, $d_j = 1$ and hence, by Lemma 3.1, the result follows. ■

Theorem 3.4 *If H be a non-semisimple Hopf algebra with antipode S of dimension p^2 over k, then $o(S^2) = p$ and hence H is isomorphic to a Taft algebra.*

Proof. By Lemma 3.3,

$$H = \bigoplus_{i,j \in \mathbf{Z}_p} H^{\omega}_{0,i,j}$$

where $\omega \in k$ is a primitive pth root of 1. Therefore, $S^{2p}(u) = u$ for all $u \in H$. By Lemma 1.3, $o(S^2) = p$. It follows from [AS98a] that H is isomorphic to a Taft algebra. ■

Acknowledgement

The author would like to thank Susan Montgomery for bringing his attention to the question on the Hopf algebras of dimension p^2.

References

[AN01] Nicolás Andruskiewitsch and Sonia Natale. Counting arguments for Hopf algebras of low dimension. *Tsukuba J. Math.*, 25(1):187–201, 2001.

[AS98a] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Hopf algebras of order p^2 and braided Hopf algebras of order p. *J. Algebra*, 199(2):430–454, 1998.

[AS98b] N. Andruskiewitsch and H.-J. Schneider. Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. *J. Algebra*, 209(2):658–691, 1998.

[BDG99] M. Beattie, S. Dăscălescu, and L. Grünenfelder. On the number of types of finite-dimensional Hopf algebras. *Invent. Math.*, 136(1):1–7, 1999.

[EG98] Pavel Etingof and Shlomo Gelaki. Semisimple Hopf algebras of dimension pq are trivial. *J. Algebra*, 210(2):664–669, 1998.

[Gel98] Shlomo Gelaki. Pointed Hopf algebras and Kaplansky's 10th conjecture. *J. Algebra*, 209(2):635–657, 1998.

[Kap75] Irving Kaplansky. *Bialgebras*. Department of Mathematics, University of Chicago, Chicago, Ill., 1975. Lecture Notes in Mathematics.

[LR87] Richard G. Larson and David E. Radford. Semisimple cosemisimple Hopf algebras. *Amer. J. Math.*, 109(1):187–195, 1987.

[LR88] Richard G. Larson and David E. Radford. Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. *J. Algebra*, 117(2):267–289, 1988.

[LR95] Richard G. Larson and David E. Radford. Semisimple Hopf algebras. *J. Algebra*, 171(1):5–35, 1995.

[Mas96] Akira Masuoka. The p^n theorem for semisimple Hopf algebras. *Proc. Amer. Math. Soc.*, 124(3):735–737, 1996.

[Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[Mon98] Susan Montgomery. Classifying finite-dimensional semisimple Hopf algebras. In *Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997)*, pages 265–279. Amer. Math. Soc., Providence, RI, 1998.

[Ng02] Siu-Hung Ng. Non-semisimple Hopf algebras of dimension p^2. *J. Algebra*, 255(1):182–197, 2002.

[NZ89] Warren D. Nichols and M. Bettina Zoeller. A Hopf algebra freeness theorem. *Amer. J. Math.*, 111(2):381–385, 1989.

[Rad76] David E. Radford. The order of the antipode of a finite dimensional Hopf algebra is finite. *Amer. J. Math.*, 98(2):333–355, 1976.

[Şte97] D. Ştefan. Hopf subalgebras of pointed Hopf algebras and applications. *Proc. Amer. Math. Soc.*, 125(11):3191–3193, 1997.

[Swe69] Moss E. Sweedler. *Hopf algebras*. W. A. Benjamin, Inc., New York, 1969. Mathematics Lecture Note Series.

[Taf71] Earl J. Taft. The order of the antipode of finite-dimensional Hopf algebra. *Proc. Nat. Acad. Sci. U.S.A.*, 68:2631–2633, 1971.

[Zhu94] Yongchang Zhu. Hopf algebras of prime dimension. *Internat. Math. Res. Notices*, (1):53–59, 1994.