Math 265 Student name: KEY

Final Exam Spring 2012 Instructor & Section: ______________________

This test is closed book and closed notes. A (graphing) calculator is allowed for this test but cannot also be a communication device (i.e., your iPhone is not a calculator). Answer each question completely using exact values unless otherwise indicated. Show your work (legibly); answers without work and/or justifications will not receive credit. Place your final answer in the provided box. Each problem is worth 10 points for a total of 80 points.

$\boxed{\text{Please announce } \#2 : z(y) \to z(t)}$

1	
2	
3	
4	
5	
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Σ	

Grading philosophy
* grade consistently
* give non-trivial points for setting a problem up
* do not give points for wrong work

DO NOT BEGIN THIS TEST UNTIL INSTRUCTED TO START

1. (a) Find a direction vector for the line of intersection of the planes $x + 2y + z = 0$ and $x + y + 1 = 0$.

Plane $\longrightarrow$ normal vector

$x+2y+z=0 \qquad \langle 1,2,1\rangle$

$x+y \quad = -1 \qquad \langle 1,1,0\rangle$

line will be $\perp$ to both normal vectors, use cross product

$$\langle 1,2,1\rangle \times \langle 1,1,0\rangle = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle 0-1, 1-0, 1-2\rangle$$

$$= \langle -1,1,-1\rangle$$

suggested grading:
1pt per normal vector (x2)
2pt for taking cross product
1pt for correct answer
5pts

Answer: $\boxed{\langle -1,1,-1\rangle}$ can be parallel to this

(b) Find the equation of the plane containing the line $x = 2t$, $y = 1 - t$, $z = 3t$ and the vector found in part (a).

$x = 2t$
$y = 1 - t$
$z = 3t$

pt: $(0,1,0)$ also point on plane
direction vector: $\langle 2,-1,3\rangle$

plane contains vector $\langle -1,1,-1\rangle$ & $\langle 2,-1,3\rangle$ use cross product to find normal vector of plane

$$\langle -1,1,-1\rangle \times \langle 2,-1,3\rangle = \begin{vmatrix} i & j & k \\ -1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = \langle 3-1, -2+3, 1-2\rangle$$

$$= \underset{\langle a,b,c\rangle}{\langle 2,1,-1\rangle}$$

$$2(x-0) + 1(y-1) - 1(z-0) = 0$$

$$2x + y - 1 - z = 0$$

suggested grading:
1pt for point in plane
2pts for taking normal
2pts for correct answer
5 pts

Answer: $\boxed{2x + y - z = 1}$

2. A particle travels along the parametric curve

$$x(t) = t + \ln t, \quad y(t) = t - \ln t \quad \cancel{z(t)}\ z(t) = 7 - 4\sqrt{t}.$$

Find the distance the particle travels in the time interval $1 \leqslant t \leqslant 2$.

$$\text{Length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt \qquad 1$$

$$= \int_1^2 \sqrt{\left(1+\frac{1}{t}\right)^2 + \left(1-\frac{1}{t}\right)^2 + \left(\frac{-2}{\sqrt{t}}\right)^2}\, dt \qquad 2$$

$$= \int_1^2 \sqrt{1 + \frac{2}{t} + \frac{1}{t^2} + 1 - \frac{2}{t} + \frac{1}{t^2} + \frac{4}{t}}\, dt$$

$$= \int_1^2 \sqrt{2 + \frac{4}{t} + \frac{2}{t^2}}\, dt$$

$$= \int_1^2 \sqrt{2\left(1+\frac{1}{t}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{2}\left(1+\frac{1}{t}\right) dt \qquad 3$$

$$= \sqrt{2}\,(t + \ln t)\Big|_1^2$$

$$= \sqrt{2}\,(2 + \ln 2) - \sqrt{2}\,(1 + \cancelto{0}{\ln 1})$$

$$= \sqrt{2}\,(1 + \ln 2)$$

Answer: $\boxed{\sqrt{2}\,(1 + \ln 2)}$ 4

Suggested grading:
3pt correct formula 1
2pt plugging in functions 2
3pt getting rid of "$\sqrt{\ }$" 3
2pt correct answer 4
10 pt

3. Let $f(x, y, z) = x^2y^3z + 5z^2$.

(a) Find a *unit* vector $\mathbf{u}$ in the direction in which f increases most rapidly at the point $\mathbf{p} = (-2, 1, -1)$.

↳ property of gradient

$$\nabla f = \langle 2xy^3z,\ 3x^2y^2z,\ x^2y^3 + 10z \rangle$$

$$\nabla f(-2,1,-1) = \langle 4, -12, -6 \rangle$$ ← right direction but not unit vector

$$\|\langle 4,-12,-6\rangle\| = \sqrt{16+144+36} = \sqrt{196} = 14$$

$$\frac{\langle 4,-12,-6\rangle}{\|\langle 4,-12,-6\rangle\|} = \left\langle \frac{4}{14}, \frac{-12}{14}, \frac{-6}{14} \right\rangle = \left\langle \frac{2}{7}, \frac{-6}{7}, \frac{-3}{7} \right\rangle$$

Answer: $\boxed{\left\langle \frac{2}{7}, \frac{-6}{7}, \frac{-3}{7} \right\rangle}$

(b) What is the rate of change in this direction?

Found by magnitude of gradient $= 14$ (from above)

Suggested grading:

4pt	formula for ∇f
2pt	$\nabla f(-2,1,-1) = \langle 4,-12,-6\rangle$
2pt	answer to (a)
2pt	answer to (b)
10 pt	

Answer: $\boxed{14}$

4. Find and classify all of the critical points for the function

$$f(x, y) = x^4 - 4xy - 7x^2 + 4y^2 + 4x - 8y + 20.$$

$$\frac{\partial f}{\partial x} = 4x^3 - 4y - 14x + 4 = 0$$

$$\frac{\partial f}{\partial y} = -4x + 8y - 8 = 0 \rightarrow 8y = 4x + 8 \rightarrow y = \tfrac{1}{2}x + 1$$

substitute $y = \frac{1}{2}x + 1$ into $\frac{\partial f}{\partial x} = 0$

$$4x^3 - 4(\tfrac{1}{2}x + 1) - 14x + 4 = 0$$

$$4x^3 - 2x - 4 - 14x + 4 = 0$$

$$4x^3 - 16x = 0$$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x(x-2)(x+2) = 0$$

$$x = 0, 2, -2$$

$x = 0$: $y = \frac{1}{2} \cdot 0 + 1 = 1$ $(0,1)$

$x = 2$: $y = \frac{1}{2} \cdot 2 + 1 = 2$ $(2,2)$

$x = -2$: $y = \frac{1}{2} \cdot (-2) + 1 = 0$ $(-2,0)$

critical pts

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$= (12x^2 - 14) \cdot 8 - (-4)^2 = (12x^2 - 14) \cdot 8 - 16$$

$D(0,1) < 0$ so saddle

$D(2,2) > 0$ and $f_{yy} = 8 > 0$ so min

$D(-2,0) > 0$ and $f_{yy} = 8 > 0$ so min

Suggested grading:

1 pt $\frac{\partial f}{\partial x} = 0$ correct partial derivatives

1 pt $\frac{\partial f}{\partial y} = 0$

3 pt correct critical pts, 1 pt each

2 pt correct formula for D

3 pt correct classification, 1 pt each

10 pt

Answer:

$(0,1)$ — SADDLE

$(2,2)$ — LOCAL MIN

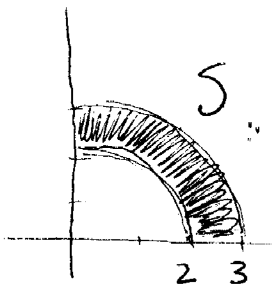
$(-2,0)$ — LOCAL MIN

5. Evaluate

$$\iint_S xy\, dA,$$

where S is the region in the first quadrant inside $x^2+y^2=9$ and outside $x^2+y^2=4$.

circles, switch to polar



$0 \le \theta \le \frac{\pi}{2}$

$2 \le r \le 3$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$\iint_S xy\, dA = \int_0^{\frac{\pi}{2}}\int_2^3 (r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_2^3 \sin\theta\cos\theta\, r^3\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \left.\frac{r^4}{4}\right|_{r=2}^{r=3} d\theta$$

$$\frac{3^4-2^4}{4} = \frac{81-16}{4} = \frac{65}{4}$$

$$= \frac{65}{4}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{65}{4}\int_0^1 u\,du$$

$$= \frac{65}{8} u^2 \Big|_0^1$$

$$= \frac{65}{8}$$

suggested grading:

4 pts describing S in polar coordinates

1 pt rewriting xy in polar

1 pt $dA = r\,dr\,d\theta$

2 pt $\int dr$

2 pt $\int d\theta$

10 pt

Answer: $\boxed{\frac{65}{8}}$

6. Find

$$\int_0^2 \int_{x-x^2}^{2-x^2} 6x\cos\left((x^2+y)^3\right)\,dy\,dx$$

by making the substitutions $u = x^2 + y$ and $v = x$.

1. $\left.\begin{array}{l} u = x^2+y \\ v = x \end{array}\right\} \longrightarrow \begin{array}{l} x = v \\ y = u - v^2 \end{array}$

2. $J(u,v) = \begin{vmatrix} 0 & 1 \\ 1 & -2v \end{vmatrix} = -1$ so $|J(u,v)| = 1$

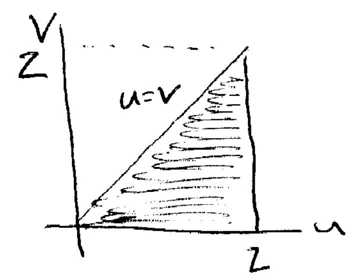
3. $6x\cos((x^2+y)^3) \longrightarrow 6v\cos(u^3)$

4. Bounds: $x=0$ to $x=2 \longrightarrow v=0$ to $v=2$

$y = x-x^2$ to $y = 2-x^2$

$x^2+y = x$ to $x^2+y=2 \longrightarrow u = v$ to $u = 2$

$$\int_0^2\int_{x-x^2}^{2-x^2} 6x\cos((x^2+y)^3)\,dy\,dx \overset{5}{=} \int_0^2\int_0^u 6v\cos(u^3)\,dv\,du$$



$$= \int_0^2 3v^2\cos(u^3)\Big|_{v=0}^{v=u}\,du$$

$$= \int_0^2 3u^2\cos(u^3)\,du$$

$$= \sin(u^3)\Big|_{u=0}^{u=2}$$

$$= \sin(8)$$

suggested grading:

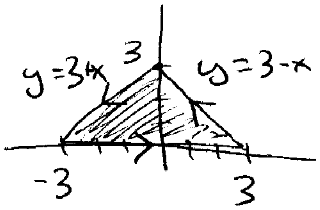
2pt (1)
1 pt (2)
1 pt (3)
3 pt (4)
1 pt (5)
2 pt doing integral correctly
10 pt

Answer: $\sin(8)$

Green's Theorem $\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

7. Let C be the closed curve which consists of straight line segments between the points $(-3,0)$, $(3,0)$ and $(0,3)$ where C is oriented counterclockwise. Find

$$\oint_C \left(y^2 + \cos(x^3) - x^2 y\right) dx + \left(2xy + \frac{1}{1+e^{2y}}\right) dy.$$



$$\oint_C \underbrace{\left(y^2+\cos(x^3)-x^2y\right)}_{M}dx + \underbrace{\left(2xy+\frac{1}{1+e^{2y}}\right)}_{N}dy$$

$$= \iint_S \left((2y) - (2y - x^2)\right) dA$$

$$= \iint_S x^2\, dA$$

$\times 2$ by symmetry

$$= 2\int_0^3 \int_0^{3-x} x^2\, dy\, dx$$

$$= 2\int_0^3 x^2 y \Big|_{y=0}^{y=3-x} dx$$

$$= 2\int_0^3 x^2(3-x)\, dx$$

$$= 2\int_0^3 (3x^2 - x^3)\, dx$$

$$= 2\left(x^3 - \frac{x^4}{4}\right)\Big|_{x=0}^{x=3}$$

$$= 2\left(3^3 - \frac{3^4}{4}\right)$$

$$= 2\left(\frac{108}{4} - \frac{81}{4}\right)$$

$$= \frac{27}{2}$$

suggested grading:
- 2pt identify/sketch region
- 4pt correctly apply Green's thm
- 4pts do the integration

10pt

Answer: $\boxed{\dfrac{27}{2}}$

spherical coordinates

8. Let $\mathcal{S}$ be the sphere of radius 2 and $\mathbf{F}(x, y, z) = \langle 3xy^2 + e^z y, 3x^2y - 3\cos(zx^3), z^3 + 17x^{25} \rangle$. Use Gauss's Divergence Theorem to calculate

$$\iint_{\partial \mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS. = \iiint_{\mathcal{S}} \nabla \cdot \mathbf{F}\, dV$$

$$\operatorname{div} F = \nabla \cdot F = 3y^2 + 3x^2 + 3z^2$$

$$\iint_{\partial S} F \cdot n\, dS = \iiint_{S} \nabla \cdot F\, dV = \iiint_{S} 3(\underbrace{x^2+y^2+z^2}_{= \rho^2})\, dV$$

$$= \underbrace{\int_0^{2\pi}\int_0^{\pi}\int_0^{2}}_{\text{bounds for sphere of radius 2}} 3\rho^2 \underbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$$

$$= \frac{3}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, \rho^5 \Big|_{\rho=0}^{\rho=2} d\phi\, d\theta$$

$$= \frac{96}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta$$

$$= \frac{96}{5}\int_0^{2\pi} -\cos\phi \Big|_0^{\pi} d\theta \qquad -(-1)-(-1) = 2$$

$$= \frac{192}{5}\int_0^{2\pi} d\theta$$

$$= \frac{384\pi}{5}$$

Answer: $\boxed{\frac{384\pi}{5}}$

Suggested grading:

3pt correctly applies divergence

4pt setting up in spherical coord.

3pt working out integral

10 pt