

Relation Algebras for Reasoning about Time and Space

Roger D. Maddux

Department of Mathematics, Iowa State University

Ames, Iowa U.S.A.

Abstract

This paper presents a brief introduction to relation algebras, including some examples motivated by work in computer science, namely, the ‘interval algebras’, relation algebras that arose from James Allen’s work on temporal reasoning, and by ‘compass algebras’, which are designed for similar reasoning about space. One kind of reasoning problem, called a ‘constraint satisfaction problem’, can be defined for arbitrary relation algebras. It will be shown here that the constraint satisfiability problem is NP-complete for almost all compass and interval algebras.

1. Relation Algebras

Composition of binary relations was introduced to logic by Augustus De Morgan [28] [29] (see [30, pp. 55–57, 208, 221, *etc.*]). De Morgan observed that the syllogism “every A is a B , every B is a C , so every A is a C ” remains valid if the copula “is” is replaced by any transitive relation L . De Morgan went further, noting that if LM is the composition of the relation L with the relation M , that is, A is an LM of B just in case A is an L of an M of B , then the following syllogism is valid: “if every A is an L of a B , and every B is an M of a C , then every A is an LM of a C .” De Morgan [29] (see [30, p. 222]) denoted the converse of the relation L by L^{-1} and its contrary by $\text{not-}L$, and observed that these operations commute: the converse of the contrary of L is the contrary of the converse of L . Around the same time, George Boole [7] [6] created algebra from the logic of classes. Starting with [31] in 1870, Charles Sanders Peirce applied Boole’s ideas to create algebra from De Morgan’s logic of relations, “and after many attempts produced a good general algebra of logic, together with another algebra specially adapted to dyadic relations (*Studies in Logic*, by members of the Johns Hopkins University, 1883, Note B, 187–203). Schröder developed the last in a systematic manner” in [35] (quotation from [26]). Peirce [32] laid out his calculus in 17 pages. F. W. K. Ernst Schröder’s investigation [35] extended this to 649 pages. His book remains today the only exhaustive treatise on the calculus of relations. For additional survey and historical material on relation algebras see [8], [11], [12], [13], [14], [20], [21], [22], [23], [24], [37], and [38].

Consider an arbitrary class, called the ‘universe of discourse’ or simply the ‘universe’. The universe could, depending on the situation and purposes, contain all possible mathematical objects, or all states of a machine, or all real numbers, or just a finite set of letters. The fundamental operations of the calculus of relations are natural set-theoretical operations on binary relations over the universe. In addition to the Boolean operations of union, intersection, and complementation, there are the ‘relative’ (as Peirce calls them), or ‘Peircean’ (as Tarski calls them) operations, namely the binary operation of ‘relative addition’ (Peirce’s name), the binary operation of ‘relative multiplication’ (Peirce’s name) or ‘composition’ (De Morgan’s name) and the unary operation of conversion. There are also four distinguished relations, namely the universal relation, the empty relation, the identity relation, and the diversity relation. The definitions of these operations and distinguished relations are listed below. In these definitions, x and y are arbitrary binary relations on the universe. By a *binary relation* we simply mean a class of ordered pairs. The ordered pair whose first element is p and whose second element is q is denoted $\langle p, q \rangle$. Thus p is an x of q if and only if $\langle p, q \rangle \in x$.

$$\begin{aligned}
x + y &= \text{union of } x \text{ and } y = \{\langle p, q \rangle : \langle p, q \rangle \in x \text{ or } \langle p, q \rangle \in y\} \\
x \cdot y &= \text{intersection of } x \text{ and } y = \{\langle p, q \rangle : \langle p, q \rangle \in x \text{ and } \langle p, q \rangle \in y\} \\
\bar{x} &= \text{complement of } x \\
&= \{\langle p, q \rangle : p, q \text{ are in the universe, but } \langle p, q \rangle \notin x\} \\
x \dagger y &= \text{relative sum of } x \text{ and } y \\
&= \{\langle p, r \rangle : \text{for every } q \text{ in the universe, } \langle p, q \rangle \in x \text{ or } \langle q, r \rangle \in y\} \\
x; y &= \text{relative product of } x \text{ and } y \\
&= \{\langle p, r \rangle : \text{for some } q, \langle p, q \rangle \in x \text{ and } \langle q, r \rangle \in y\} \\
\check{x} &= \text{converse of } x = \{\langle q, p \rangle : \langle p, q \rangle \in x\} \\
1 &= \text{universal relation} = \{\langle p, q \rangle : p, q \text{ are in the universe}\} \\
0 &= \text{empty relation} = \emptyset \\
1' &= \text{identity relation} = \{\langle p, p \rangle : p \text{ is in the universe}\} \\
0' &= \text{diversity relation} = \{\langle p, q \rangle : p, q \text{ are in the universe, } p \neq q\}
\end{aligned}$$

Both De Morgan and Peirce denoted the composition of x and y simply by “ xy ”, but Schröder [35] used “ $x;y$ ”, as is done here. The notation “ $x|y$ ” was used by Whitehead and Russell [45] and adopted by Tarski and his school [10]. Peirce introduced the notation “ \check{x} ” for the converse of x . Schröder introduced “ $1'$ ” and “ $0'$ ” for the identity and diversity relations. Here are some laws in the calculus of relations. These laws hold for every possible universe, and all possible binary relations x , y , and z .

$$\begin{aligned}
\text{(i)} \quad & (x + y) + z = x + (y + z) \\
\text{(ii)} \quad & x + y = y + x \\
\text{(iii)} \quad & x = \overline{\bar{x} + y} + \overline{\bar{x} + \bar{y}}
\end{aligned}$$

- (iv) $x \cdot y = \overline{\overline{x} + \overline{y}}$
- (v) $1 = x + \overline{x}$
- (vi) $0 = \overline{1}$
- (vii) $x;(y;z) = (x;y);z$
- (viii) $x;1' = x$
- (ix) $(x + y);z = x;z + y;z$
- (x) $\check{x} = x$
- (xi) $(x + y)^\vee = \check{x} + \check{y}$
- (xii) $(x;y)^\vee = \check{y};\check{x}$
- (xiii) $\check{x};\overline{\check{x};\overline{y}} + \overline{y} = \overline{y}$
- (xiv) $0' = \overline{1}$
- (xv) $x \dagger y = \overline{\overline{x};\overline{y}}$

A *relation algebra* is an algebra of the form $\mathfrak{A} = \langle \mathfrak{A}, +, \cdot, \overline{}, \mathbf{o}, \mathbf{1}, \dagger, ;, \check{}, \vee, \mathbf{o}', \mathbf{1}' \rangle$ that satisfies the identities (i)–(xv) above. Identities (i)–(vi) say that $\langle \mathfrak{A}, +, \cdot, \overline{}, \mathbf{o}, \mathbf{1} \rangle$ is a Boolean algebra (called the *Boolean part* or *Boolean reduct* of \mathfrak{A}). One of the most significant laws of the calculus of relations is De Morgan’s “Theorem K” (see [30, pp. 186–7, 224] or [24, p. 434–5]), which asserts that the following statements are equivalent: $x;y \leq z$, $\check{x};\overline{z} \leq \overline{y}$, $\overline{z};\check{y} \leq \overline{x}$. After minor Boolean transformations Theorem K becomes the *cycle law*, that the following statements are equivalent: $x;y \cdot z = 0$, $\check{x};z \cdot y = 0$, $z;\check{y} \cdot x = 0$. The cycle law and De Morgan’s Theorem K hold in every relation algebra because they can be proved from axioms (i)–(xv). There are many other equivalent axiomatizations for relation algebras. For example, equations (ix)–(xiii) can be replaced with the cycle law or with Theorem K. Also, equations (i)–(vi) can be replaced with some other set of equations that define Boolean algebras.

The algebra containing all binary relations on the universe U is denoted $\mathfrak{Rc}(U)$. Identities (i)–(xv) hold in $\mathfrak{Rc}(U)$, so $\mathfrak{Rc}(U)$ is a relation algebra. Relation algebras are defined by equations, so it follows that subalgebras, homomorphic images, and direct products of relation algebras are again relation algebras. The algebras that can be obtained from algebras of the form $\mathfrak{Rc}(U)$ by forming subalgebras, homomorphic images, and direct products are called *representable* relation algebras. Roger Lyndon [18] showed that not all relation algebras are representable. It follows that the axioms (i)–(xv) are incomplete, in the sense that there are equations which hold in every algebra of the form $\mathfrak{Rc}(U)$ but cannot be derived from (i)–(xv). J. Donald Monk [27] proved that the equations that hold in every algebra of the form $\mathfrak{Rc}(U)$ does not have a finite axiomatization.

Let \mathfrak{A} be a relation algebra. An element x of \mathfrak{A} is an *atom* if x is not 0 and there is no other element between x and 0, that is, either $x \cdot y = x$ or $x \cdot y = 0$ for every y in \mathfrak{A} . Let $At\mathfrak{A}$ be the set of atoms of \mathfrak{A} . It is easy to prove that the converse of an atom is an atom, *i.e.*, if $x \in At\mathfrak{A}$ then $\check{x} \in At\mathfrak{A}$. The relation algebra \mathfrak{A} is said to be *atomic* if there is an atom below every nonzero element, that is, if $y \neq 0$ then there is some $x \in At\mathfrak{A}$ such that $x \leq y$. \mathfrak{A} is said to

be *complete* if every subset X of \mathfrak{A} has a least upper bound $\sum X$ and greatest lower bound $\prod X$.

It turns out that if \mathfrak{A} is both complete and atomic, then the structure of \mathfrak{A} is entirely determined by its atoms and the action of the relative operations on the atoms. For a precise statement of this fact, let $C(\mathfrak{A}) = \{\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle : \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{At}\mathfrak{A} \text{ and } \mathbf{a}; \mathbf{b} \geq \mathbf{c}\}$ and $I(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \in \mathfrak{At}\mathfrak{A} \text{ and } \mathbf{a} \leq \mathbf{1}'\}$. $C(\mathfrak{A})$ is the set of *cycles* of \mathfrak{A} and $I(\mathfrak{A})$ is the set of *identity atoms* of \mathfrak{A} . Define the *atom structure* of \mathfrak{A} to be $\mathfrak{At}\mathfrak{A} = \langle \mathfrak{At}\mathfrak{A}, \mathfrak{C}(\mathfrak{A}), \check{\cdot}, \mathfrak{I}(\mathfrak{A}) \rangle$. Any two complete atomic relation algebras with the isomorphic atom structures are isomorphic. For any $a, b, c \in \mathfrak{At}\mathfrak{A}$, let $[a, b, c] = \{\langle a, b, c \rangle, \langle \check{a}, c, b \rangle, \langle b, \check{c}, \check{a} \rangle, \langle \check{b}, \check{a}, \check{c} \rangle, \langle \check{c}, a, \check{b} \rangle, \langle c, \check{b}, a \rangle\}$. By the cycle law, the set $C(\mathfrak{A})$ of cycles of \mathfrak{A} is a union of sets of the form $[a, b, c]$. We refer to such sets as *cyclesets*. The identity element, the converse of x , and the relative product of x and y can be computed from the atom structure as follows: $\mathbf{1}' = \sum I(\mathfrak{A})$, $\check{x} = \sum \{\check{a} : x \geq a \in \mathfrak{At}\mathfrak{A}\}$, and $x; y = \sum \{c : \text{there are } a, b \in \mathfrak{At}\mathfrak{A} \text{ with } a \leq x, b \leq y, \text{ and } \langle a, b, c \rangle \in C(\mathfrak{A})\}$. Hence to specify a complete atomic relation algebra it suffices to list its atoms, to list its identity atoms, to indicate the converse of each atom, and, finally, to list the cyclesets $[a, b, c]$. This is especially convenient when \mathfrak{A} is finite. We present several examples of relation algebras using this method.

2. Interval Algebras

To define the interval algebra IA [1] [2] take the universe U to be the set of all ‘events’, where an event is simply a pair of real numbers, the second of which is larger than the first. The first number in an event is its ‘starting time’, the second its ‘ending time’. (Our model for time here is just the real numbers.) Seven binary relations on events are defined in the list below, where x, x', y, y' are real numbers and $\langle x, x' \rangle, \langle y, y' \rangle$ are events.

identity:	$\mathbf{1}' = \{\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle : x = y < x' = y'\}$
precedes:	$p = \{\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle : x < x' < y < y'\}$
during:	$d = \{\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle : y < x < x' < y'\}$
overlaps:	$o = \{\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle : x < y < x' < y'\}$
meets:	$m = \{\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle : x < x' = y < y'\}$
starts:	$s = \{\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle : x = y < x' < y'\}$
finishes:	$f = \{\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle : y < x < x' = y'\}$

These seven relations are studied in [42] and are used in some computer programs [5] [25] [36]. They generate a finite subalgebra of $\mathfrak{Re}(\mathfrak{A})$, called the *interval algebra*, or simply the IA. The IA has 13 atoms, namely $\mathbf{1}'$, p , \check{p} , d , \check{d} , o , \check{o} , m , \check{m} , s , \check{s} , f , and \check{f} . (It turns out that p alone will generate the IA, and so will each of the elements \check{p} , m , \check{m} , o , and \check{o} [17] [16, Theorem 4.4].) If we start with the rational numbers instead of the reals, or, in fact, any dense linear ordering without endpoints, then the resulting algebra is isomorphic to

the IA. But if we use some other infinite linear ordering, then the relation algebra generated by $1', p, d, o, m, s,$ and f may be infinite. This happens, for example, when we use the integers. If we start with a finite linear ordering on U , then the subalgebra generated by $1', p, d, o, m, s,$ and f will be $\mathfrak{Re}(\mathfrak{U})$. Any relation algebra obtained in this way will be called *an* interval algebra (while *the* IA is the one obtained from the reals or rationals). The IA has 75 cyclesets: $[1', 1', 1'], [1', s, s], [1', m, m], [1', p, p], [1', o, o], [1', f, f], [1', d, d], [s, 1', s], [s, s, s], [s, m, p], [s, p, p], [s, o, m], [s, o, p], [s, o, o], [s, f, d], [s, d, d], [m, 1', m], [m, s, m], [m, m, p], [m, p, p], [m, o, p], [m, f, s], [m, f, o], [m, f, d], [m, d, s], [m, d, o], [m, d, d], [p, 1', p], [p, s, p], [p, m, p], [p, p, p], [p, o, p], [p, f, s], [p, f, m], [p, f, p], [p, f, o], [p, f, d], [p, d, s], [p, d, m], [p, d, p], [p, d, o], [p, d, d], [o, 1', o], [o, s, o], [o, m, p], [o, p, p], [o, o, m], [o, o, p], [o, o, o], [o, f, s], [o, f, o], [o, f, d], [o, d, s], [o, d, o], [o, d, d], [f, 1', f], [f, s, d], [f, m, m], [f, p, p], [f, o, s], [f, o, o], [f, o, d], [f, f, f], [f, d, d], [d, 1', d], [d, s, d], [d, m, p], [d, p, p], [d, o, s], [d, o, m], [d, o, p], [d, o, o], [d, o, d], [d, f, d], [d, d, d]$. Although all relative products in the IA can be computed from the cycles, it is convenient to also have the products in a table. The table of relative products of atoms of the IA is given in Figs. 1 and 2. To save space the $+$ signs are omitted, so, for example, $pdoms = p + d + o + m + s$. The table appeared first in [2]. It not only shows relative products of atoms in the IA, but also shows containments for the Allen-Hayes algebra [3] [4]. By the *Allen-Hayes algebra* we mean the direct product of ‘all’ interval algebras, *i.e.*, the direct product of an indexed system of algebras containing one algebra from each isomorphism type of interval algebras. The Allen-Hayes algebra contains the elements $1', p, \check{p}, d, \check{d}, o, \check{o}, m, \check{m}, s, \check{s}, f,$ and \check{f} . They form a partition, *i.e.*, they are pairwise disjoint and $1 = 1' + p + \check{p} + d + \check{d} + o + \check{o} + m + \check{m} + s + \check{s} + f + \check{f}$. Finally, the relative product of any two of them is contained in (and not necessarily equal to) the corresponding entry in the table.

3. Compass Algebras

Let the universe be the set of all points in the n -dimensional Euclidean space \mathbb{R}^n , where \mathbb{R} is the set of real numbers. Let \mathbb{R}^+ be the set of positive real numbers. For every vector \mathbf{v} in \mathbb{R}^n define two binary relations on \mathbb{R}^n as follows:

$$D_{\mathbf{v}} = \{ \langle \mathbf{x}, \mathbf{y} \rangle : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and for some } r \text{ in } \mathbb{R}^+, \mathbf{x} + r\mathbf{v} = \mathbf{y} \},$$

$$E_{\mathbf{v}} = \{ \langle \mathbf{x}, \mathbf{y} \rangle : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and for some } r \text{ in } \mathbb{R}, \mathbf{x} + r\mathbf{v} = \mathbf{y} \}.$$

Here are some easily proved properties of these relations.

- Theorem 1.**
- (i) $D_{\mathbf{0}} = E_{\mathbf{0}} = 1' = \{ \langle \mathbf{x}, \mathbf{x} \rangle : \mathbf{x} \in \mathbb{R}^n \},$
 - (ii) $\check{D}_{\mathbf{v}} = D_{-\mathbf{v}} = \{ \langle \mathbf{x}, \mathbf{y} \rangle : \text{for some } r \text{ in } \mathbb{R}^+, \mathbf{x} - r\mathbf{v} = \mathbf{y} \},$
 - (iii) $D_{\mathbf{v}}; D_{\mathbf{v}} = D_{\mathbf{v}},$
 - (iv) $D_{\mathbf{v}} = D_{r\mathbf{v}}$ and $E_{\mathbf{v}} = E_{r\mathbf{v}}$ whenever $r \in \mathbb{R}^+,$
 - (v) $E_{\mathbf{v}} = \check{E}_{\mathbf{v}} = E_{\mathbf{v}}; E_{\mathbf{v}} = D_{\mathbf{v}}; \check{D}_{\mathbf{v}} = \check{D}_{\mathbf{v}}; D_{\mathbf{v}} = D_{\mathbf{v}} + \check{D}_{\mathbf{v}} + D_{\mathbf{0}},$
 - (vi) $E_{\mathbf{v}}$ is an equivalence relation on $\mathbb{R}^n,$

	1'	<i>p</i>	<i>ř</i>	<i>d</i>	<i>ď</i>	<i>o</i>	<i>ů</i>
1'	1'	<i>p</i>	<i>ř</i>	<i>d</i>	<i>ď</i>	<i>o</i>	<i>ů</i>
<i>p</i>	<i>p</i>	<i>p</i>	1	<i>pdoms</i>	<i>p</i>	<i>p</i>	<i>pdoms</i>
<i>ř</i>	<i>ř</i>	1	<i>ř</i>	<i>řdřm.f</i>	<i>ř</i>	<i>řdřm.f</i>	<i>ř</i>
<i>d</i>	<i>d</i>	<i>p</i>	<i>ř</i>	<i>d</i>	1	<i>pdoms</i>	<i>řdřm.f</i>
<i>ď</i>	<i>ď</i>	<i>přom.ř</i>	<i>řdřm.ř</i>	1' <i>ddřřřřřřřř</i>	<i>ď</i>	<i>ďoř</i>	<i>ďřř</i>
<i>o</i>	<i>o</i>	<i>p</i>	<i>řdřm.ř</i>	<i>dos</i>	<i>přom.ř</i>	<i>pom</i>	1' <i>ddřřřřřřřř</i>
<i>ů</i>	<i>ů</i>	<i>přom.ř</i>	<i>ř</i>	<i>ďoř</i>	<i>řdřm.ř</i>	1' <i>ddřřřřřřřř</i>	<i>řřm</i>
<i>m</i>	<i>m</i>	<i>p</i>	<i>řdřm.ř</i>	<i>dos</i>	<i>p</i>	<i>p</i>	<i>dos</i>
<i>řm</i>	<i>řm</i>	<i>přom.ř</i>	<i>ř</i>	<i>ďoř</i>	<i>ř</i>	<i>ďoř</i>	<i>ř</i>
<i>s</i>	<i>s</i>	<i>p</i>	<i>ř</i>	<i>d</i>	<i>přom.ř</i>	<i>pom</i>	<i>ďoř</i>
<i>ř</i>	<i>ř</i>	<i>přom.ř</i>	<i>ř</i>	<i>ďoř</i>	<i>ď</i>	<i>ďoř</i>	<i>ů</i>
<i>f</i>	<i>f</i>	<i>p</i>	<i>ř</i>	<i>d</i>	<i>řdřm.ř</i>	<i>dos</i>	<i>řřm</i>
<i>ř</i>	<i>ř</i>	<i>p</i>	<i>řdřm.ř</i>	<i>dos</i>	<i>ď</i>	<i>o</i>	<i>ďřř</i>

FIGURE 1. Products for the interval algebra, first part.

	<i>m</i>	<i>řm</i>	<i>s</i>	<i>ř</i>	<i>f</i>	<i>ř</i>
1'	<i>m</i>	<i>řm</i>	<i>s</i>	<i>ř</i>	<i>f</i>	<i>ř</i>
<i>p</i>	<i>p</i>	<i>pdoms</i>	<i>p</i>	<i>p</i>	<i>pdoms</i>	<i>p</i>
<i>ř</i>	<i>řdřm.f</i>	<i>ř</i>	<i>řdřm.f</i>	<i>ř</i>	<i>ř</i>	<i>ř</i>
<i>d</i>	<i>p</i>	<i>ř</i>	<i>d</i>	<i>řdřm.f</i>	<i>d</i>	<i>pdoms</i>
<i>ď</i>	<i>ďoř</i>	<i>ďřř</i>	<i>ďoř</i>	<i>ď</i>	<i>ďřř</i>	<i>ď</i>
<i>o</i>	<i>p</i>	<i>ďřř</i>	<i>o</i>	<i>ďoř</i>	<i>dos</i>	<i>pom</i>
<i>ů</i>	<i>ďoř</i>	<i>ř</i>	<i>ďoř</i>	<i>řřm</i>	<i>ů</i>	<i>ďřř</i>
<i>m</i>	<i>p</i>	1' <i>fř</i>	<i>m</i>	<i>m</i>	<i>dos</i>	<i>p</i>
<i>řm</i>	1' <i>sř</i>	<i>ř</i>	<i>ďoř</i>	<i>ř</i>	<i>řm</i>	<i>řm</i>
<i>s</i>	<i>p</i>	<i>řm</i>	<i>s</i>	1' <i>sř</i>	<i>d</i>	<i>pom</i>
<i>ř</i>	<i>ďoř</i>	<i>řm</i>	1' <i>sř</i>	<i>ř</i>	<i>ů</i>	<i>ď</i>
<i>f</i>	<i>m</i>	<i>ř</i>	<i>d</i>	<i>řřm</i>	<i>f</i>	1' <i>fř</i>
<i>ř</i>	<i>m</i>	<i>ďřř</i>	<i>o</i>	<i>ď</i>	1' <i>fř</i>	<i>ř</i>

FIGURE 2. Products for interval algebra, second part.

	1'	a	ǎ
1'	1'	a	ǎ
a	a	a	1
ǎ	ǎ	1	a

FIGURE 3. Products for $\mathfrak{C}_1[\langle 1 \rangle]$

- (vii) $D_{\mathbf{v}}; D_{\mathbf{w}} = D_{\mathbf{w}}; D_{\mathbf{v}} = \{\langle \mathbf{x}, \mathbf{y} \rangle : \text{for some } r, s \text{ in } \mathbb{R}^+, \mathbf{x} + r\mathbf{v} + s\mathbf{w} = \mathbf{y}\}$,
- (viii) $E_{\mathbf{v}}; E_{\mathbf{w}} = E_{\mathbf{w}}; E_{\mathbf{v}} = \{\langle \mathbf{x}, \mathbf{y} \rangle : \text{for some } r, s \text{ in } \mathbb{R}, \mathbf{x} + r\mathbf{v} + s\mathbf{w} = \mathbf{y}\}$,
- (ix) $\langle \mathbf{x}, \mathbf{y} \rangle \in E_{\mathbf{v}}$ if and only if $\mathbf{y} - \mathbf{x}$ is in the subspace spanned by \mathbf{v} ,
- (x) $\langle \mathbf{x}, \mathbf{y} \rangle \in E_{\mathbf{v}}; E_{\mathbf{w}}$ if and only if $\mathbf{y} - \mathbf{x}$ is in the subspace spanned by \mathbf{v} and \mathbf{w} ,
- (xi) $\langle \mathbf{x}, \mathbf{y} \rangle \in E_{\mathbf{v}_1}; \dots; E_{\mathbf{v}_m}$ if and only if $\mathbf{y} - \mathbf{x}$ is in the subspace spanned by $\mathbf{v}_1, \dots, \mathbf{v}_m$.

For any m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^n$, let $\mathfrak{C}_n[\mathbf{v}_1, \dots, \mathbf{v}_m]$ be the subalgebra of $\mathfrak{Re}(\mathbb{R}^n)$ generated by the relations $D_{\mathbf{v}_1}, \dots, D_{\mathbf{v}_m}$. $\mathfrak{C}_n[\mathbf{v}_1, \dots, \mathbf{v}_m]$ is called the n -dimensional compass algebra determined by $\mathbf{v}_1, \dots, \mathbf{v}_m$. If \mathbf{v} and \mathbf{w} are a linearly dependent pair of nonzero vectors, then either $D_{\mathbf{v}} = D_{\mathbf{w}}$ or $D_{\mathbf{v}} = \check{D}_{\mathbf{w}}$. If \mathbf{v} and \mathbf{w} both appear in a list of vectors generating a compass algebra, then \mathbf{v} can be deleted from the list, and the same compass algebra will still be obtained from the remaining vectors. Even if the vectors are pairwise linearly independent, deleting one of them may not result in a strictly smaller compass algebra. The structure of $\mathfrak{C}_n[\mathbf{v}_1, \dots, \mathbf{v}_m]$ depends heavily on the choice of vectors. But if $\mathbf{v}_1, \dots, \mathbf{v}_m$ is a linear independent set of vectors, then the structure of $\mathfrak{C}_n[\mathbf{v}_1, \dots, \mathbf{v}_m]$ is completely determined by m . More exactly, if $\mathbf{v}_1, \dots, \mathbf{v}_m$ and $\mathbf{v}'_1, \dots, \mathbf{v}'_m$ are two linearly independent sets of vectors in \mathbb{R}^n (hence $m \leq n$), then $\mathfrak{C}_n[\mathbf{v}_1, \dots, \mathbf{v}_m]$ is isomorphic to $\mathfrak{C}_n[\mathbf{v}'_1, \dots, \mathbf{v}'_m]$.

The 1-dimensional compass algebra $\mathfrak{C}_1[\langle 1 \rangle]$ generated by the 1-dimensional vector $\langle 1 \rangle$ has three atoms, namely $D_{\langle 1 \rangle}$, $D_{\langle -1 \rangle}$, and $D_{\langle 0 \rangle}$. $\mathfrak{C}_1[\langle 1 \rangle]$ is known as the ‘Point Algebra’ [15] [17] [16] [39] [40] [41] [43] [44]. For a description of the structure of $\mathfrak{C}_1[\langle 1 \rangle]$ in terms of atoms and cycles, let $1' = D_{\langle 0 \rangle}$, $a = D_{\langle 1 \rangle}$, and $\check{a} = D_{\langle -1 \rangle}$. Then the cyclesets of \mathfrak{A} are $[1', 1', 1']$, $[1', a, a]$, $[a, 1', a]$, and $[a, a, a]$. The table of relative products of atoms is in Fig. 3. Every 1-dimensional vector in 1-space must determine one of the relations $D_{\langle 1 \rangle}$, $D_{\langle -1 \rangle}$, or $D_{\langle 0 \rangle}$, so no new 1-dimensional compass algebras are obtained by considering two or more vectors in 1-dimensional space. However, there is one other 1-dimensional compass algebra, namely $\mathfrak{C}_1[\langle 0 \rangle]$. This algebra has two atoms, namely $D_{\langle 0 \rangle} = 1'$ and $D_{\langle 1 \rangle} + D_{\langle -1 \rangle} = 0'$. The cyclesets of \mathfrak{A} are $[1', 1', 1']$, $[1', 0', 0']$, and $[0', 0', 0']$, and the table of relative products of atoms is in Fig. 4. By comparing this and the previous table it can be seen that $\mathfrak{C}_1[\langle 0 \rangle]$ is isomorphic to a subalgebra of $\mathfrak{C}_1[\langle 1 \rangle]$, the one with atoms $1'$ and $a + \check{a}$. Also, $\mathfrak{C}_1[\langle 0 \rangle]$ is isomorphic to $\mathfrak{C}_n[\langle 0 \rangle]$ for every integer n .

	1'	0'
1'	1'	0'
0'	0'	1

FIGURE 4. Products for $\mathfrak{C}_1[\langle \mathbf{o} \rangle]$

Now we consider 2-dimensional compass algebras. Among these are particular algebras which inspired the name ‘‘compass algebra’’. We start with the compass algebra $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle, \langle \mathbf{o}, \mathbf{1} \rangle]$. We would get the same algebra with any two linearly independent vectors in \mathbb{R}^2 , but these two allow us to dub $E_{\langle \mathbf{1}, \mathbf{o} \rangle}$ the ‘east-west’ direction, while $E_{\langle \mathbf{o}, \mathbf{1} \rangle}$ is the ‘north-south’ direction. Thus $\mathfrak{C}_2[\langle \mathbf{0}, \mathbf{1} \rangle, \langle \mathbf{1}, \mathbf{0} \rangle]$ is a ‘2-directional’ algebra of relations. ‘East’, ‘west’, ‘north’, and ‘south’ are the relations $D_{\langle \mathbf{1}, \mathbf{o} \rangle}$, $D_{\langle -\mathbf{1}, \mathbf{o} \rangle}$, $D_{\langle \mathbf{o}, \mathbf{1} \rangle}$, and $D_{\langle \mathbf{o}, -\mathbf{1} \rangle}$, respectively. In the standard Euclidean plane of analytic geometry, the points ‘east’ of the origin are all the points on the positive part of the x -axis, and so on. $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle, \langle \mathbf{o}, \mathbf{1} \rangle]$ has nine atoms, namely $D_{\langle \mathbf{0}, \mathbf{0} \rangle}$, $D_{\langle \mathbf{1}, \mathbf{o} \rangle}$, $D_{\langle -\mathbf{1}, \mathbf{o} \rangle}$, $D_{\langle \mathbf{o}, \mathbf{1} \rangle}$, $D_{\langle \mathbf{o}, -\mathbf{1} \rangle}$, $D_{\langle \mathbf{o}, \mathbf{1} \rangle}; D_{\langle \mathbf{1}, \mathbf{o} \rangle}$, $D_{\langle \mathbf{o}, \mathbf{1} \rangle}; D_{\langle -\mathbf{1}, \mathbf{o} \rangle}$, $D_{\langle \mathbf{o}, -\mathbf{1} \rangle}; D_{\langle \mathbf{1}, \mathbf{o} \rangle}$, and $D_{\langle \mathbf{o}, -\mathbf{1} \rangle}; D_{\langle \mathbf{o}, -\mathbf{1} \rangle}$. The last four atoms could be called ‘northeasterly’, ‘northwesterly’, ‘southeasterly’, and ‘southwesterly’, respectively, since they do not correspond exactly with directions of the compass. The points in the Euclidean plane which can be reached by going north-easterly from the origin are exactly those in the first quadrant. Let

$$\begin{aligned}
1' &= D_{\langle \mathbf{0}, \mathbf{0} \rangle} = \text{identity} \\
a &= D_{\langle \mathbf{1}, \mathbf{o} \rangle} = \text{east} & b &= a; c = c; a = \text{northeasterly} \\
\check{a} &= D_{\langle -\mathbf{1}, \mathbf{o} \rangle} = \text{west} & \check{b} &= \check{c}; \check{a} = \check{a}; \check{c} = \text{southwesterly} \\
c &= D_{\langle \mathbf{o}, \mathbf{1} \rangle} = \text{north} & d &= \check{a}; c = c; \check{a} = \text{northwesterly} \\
\check{c} &= D_{\langle \mathbf{o}, -\mathbf{1} \rangle} = \text{south} & \check{d} &= \check{c}; a = a; \check{c} = \text{southeasterly}
\end{aligned}$$

Then the 33 cyclesets of $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle, \langle \mathbf{o}, \mathbf{1} \rangle]$ are $[1', 1', 1']$, $[1', a, a]$, $[a, 1', a]$, $[1', b, b]$, $[b, 1', b]$, $[1', c, c]$, $[c, 1', c]$, $[1', d, d]$, $[d, 1', d]$, $[a, a, a]$, $[a, b, b]$, $[a, c, b]$, $[a, d, b]$, $[a, d, c]$, $[a, d, d]$, $[b, a, b]$, $[b, b, b]$, $[b, c, b]$, $[b, d, b]$, $[b, d, c]$, $[b, d, d]$, $[c, a, b]$, $[c, b, b]$, $[c, c, c]$, $[c, d, d]$, $[d, a, b]$, $[d, a, c]$, $[d, a, d]$, $[d, b, b]$, $[d, b, c]$, $[d, b, d]$, $[d, c, d]$, $[d, d, d]$. The relative products of atoms are given in Fig. 5. The compass algebra $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle, \langle \mathbf{1}, \mathbf{1} \rangle, \langle \mathbf{o}, \mathbf{1} \rangle]$ has 13 atoms, namely $1'$, a , b , c , d , e , f , \check{a} , \check{b} , \check{c} , \check{d} , \check{e} , and \check{f} , where $1' = D_{\langle \mathbf{0}, \mathbf{0} \rangle}$, $a = D_{\langle \mathbf{1}, \mathbf{o} \rangle}$, $b = D_{\langle \mathbf{1}, \mathbf{o} \rangle}; D_{\langle \mathbf{1}, \mathbf{1} \rangle}$, $c = D_{\langle \mathbf{1}, \mathbf{1} \rangle}$, $d = D_{\langle \mathbf{1}, \mathbf{1} \rangle}; D_{\langle \mathbf{o}, \mathbf{1} \rangle}$, $e = D_{\langle \mathbf{o}, \mathbf{1} \rangle}$, and $f = D_{\langle \mathbf{o}, \mathbf{1} \rangle}; D_{\langle -\mathbf{1}, \mathbf{o} \rangle}$. There are 89 cyclesets, each having the form $[x, y, z]$ with x, y, z in $\{1', a, b, c, d, e, f\}$. The cyclesets are not listed, but they can be read from the tables of relative products in Figs. 6, 7.

Set $\hat{x} = x + \check{x}$ for every x in $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle, \langle \mathbf{1}, \mathbf{1} \rangle, \langle \mathbf{o}, \mathbf{1} \rangle]$. Then $1' + \hat{a} = E_{\langle \mathbf{1}, \mathbf{o} \rangle}$, $1' + \hat{c} = E_{\langle \mathbf{1}, \mathbf{1} \rangle}$, $1' + \hat{e} = E_{\langle \mathbf{o}, \mathbf{1} \rangle}$, etc. and $1'$, \hat{a} , \hat{b} , \hat{c} , \hat{d} , \hat{e} , and \hat{f} are the atoms of a subalgebra called the ‘symmetric subalgebra’ of $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle, \langle \mathbf{1}, \mathbf{1} \rangle, \langle \mathbf{o}, \mathbf{1} \rangle]$. The table of products for this subalgebra is in Fig. 8. Consider the 2-dimensional com-

	1'	<i>a</i>	\check{a}	<i>b</i>	\check{b}	<i>c</i>	\check{c}	<i>d</i>	\check{d}
1'	1'	<i>a</i>	\check{a}	<i>b</i>	\check{b}	<i>c</i>	\check{c}	<i>d</i>	\check{d}
<i>a</i>	<i>a</i>	<i>a</i>	1' <i>a</i> \check{a}	<i>b</i>	$\check{b}\check{c}\check{d}$	<i>b</i>	\check{d}	<i>bcd</i>	\check{d}
<i>b</i>	<i>b</i>	<i>b</i>	<i>bcd</i>	<i>b</i>	1	<i>b</i>	<i>ab</i> \check{d}	<i>bcd</i>	<i>ab</i> \check{d}
<i>c</i>	<i>c</i>	<i>b</i>	<i>d</i>	<i>b</i>	<i>d</i> $\check{a}\check{b}$	<i>c</i>	1' <i>c</i> \check{c}	<i>d</i>	<i>ab</i> \check{d}
<i>d</i>	<i>d</i>	<i>bcd</i>	<i>d</i>	<i>bcd</i>	<i>d</i> $\check{a}\check{b}$	<i>d</i>	<i>d</i> $\check{a}\check{b}$	<i>d</i>	1
\check{a}	\check{a}	1' <i>a</i> \check{a}	\check{a}	<i>bcd</i>	\check{b}	<i>d</i>	\check{b}	<i>d</i>	$\check{b}\check{c}\check{d}$
\check{b}	\check{b}	$\check{b}\check{c}\check{d}$	\check{b}	1	\check{b}	<i>d</i> $\check{a}\check{b}$	\check{b}	<i>d</i> $\check{a}\check{b}$	$\check{b}\check{c}\check{d}$
\check{c}	\check{c}	\check{d}	\check{b}	<i>ab</i> \check{d}	\check{b}	1' <i>c</i> \check{c}	\check{c}	<i>d</i> $\check{a}\check{b}$	\check{d}
\check{d}	\check{d}	\check{d}	$\check{b}\check{c}\check{d}$	<i>ab</i> \check{d}	$\check{b}\check{c}\check{d}$	<i>ab</i> \check{d}	\check{d}	1	\check{d}

FIGURE 5. Products for $\mathfrak{C}_2[\langle 1, \mathbf{o} \rangle, \langle \mathbf{o}, 1 \rangle]$

	1'	<i>a</i>	\check{a}	<i>b</i>	\check{b}	<i>c</i>	\check{c}
1'	1'	<i>a</i>	\check{a}	<i>b</i>	\check{b}	<i>c</i>	\check{c}
<i>a</i>	<i>a</i>	<i>a</i>	1' <i>a</i> \check{a}	<i>b</i>	$\check{b}\check{c}\check{d}\check{e}\check{f}$	<i>b</i>	$\check{d}\check{e}\check{f}$
\check{a}	\check{a}	1' <i>a</i> \check{a}	\check{a}	<i>bcdef</i>	\check{b}	<i>def</i>	\check{b}
<i>b</i>	<i>b</i>	<i>b</i>	<i>bcdef</i>	<i>b</i>	1	<i>b</i>	<i>ab</i> $\check{d}\check{e}\check{f}$
\check{b}	\check{b}	$\check{b}\check{c}\check{d}\check{e}\check{f}$	\check{b}	1	\check{b}	<i>def</i> $\check{a}\check{b}$	\check{b}
<i>c</i>	<i>c</i>	<i>b</i>	<i>def</i>	<i>b</i>	<i>def</i> $\check{a}\check{b}$	<i>c</i>	1' <i>c</i> \check{c}
\check{c}	\check{c}	$\check{d}\check{e}\check{f}$	\check{b}	<i>ab</i> $\check{d}\check{e}\check{f}$	\check{b}	1' <i>c</i> \check{c}	\check{c}
<i>d</i>	<i>d</i>	<i>bcd</i>	<i>def</i>	<i>bcd</i>	<i>def</i> $\check{a}\check{b}$	<i>d</i>	<i>def</i> $\check{a}\check{b}$
\check{d}	\check{d}	$\check{d}\check{e}\check{f}$	$\check{b}\check{c}\check{d}$	<i>ab</i> $\check{d}\check{e}\check{f}$	$\check{b}\check{c}\check{d}$	<i>ab</i> $\check{d}\check{e}\check{f}$	\check{d}
<i>e</i>	<i>e</i>	<i>bcd</i>	<i>f</i>	<i>bcd</i>	<i>f</i> $\check{a}\check{b}$	<i>d</i>	<i>f</i> $\check{a}\check{b}$
\check{e}	\check{e}	\check{f}	$\check{b}\check{c}\check{d}$	<i>ab</i> \check{f}	$\check{b}\check{c}\check{d}$	<i>ab</i> \check{f}	\check{d}
<i>f</i>	<i>f</i>	<i>bcdef</i>	<i>f</i>	<i>bcdef</i>	<i>f</i> $\check{a}\check{b}$	<i>def</i>	<i>f</i> $\check{a}\check{b}$
\check{f}	\check{f}	\check{f}	$\check{b}\check{c}\check{d}\check{e}\check{f}$	<i>ab</i> \check{f}	$\check{b}\check{c}\check{d}\check{e}\check{f}$	<i>ab</i> \check{f}	$\check{d}\check{e}\check{f}$

FIGURE 6. Products for $\mathfrak{C}_2[\langle 1, \mathbf{o} \rangle, \langle 1, 1 \rangle, \langle \mathbf{o}, 1 \rangle]$, first part

pass algebra $\mathfrak{C}_2[\langle 1, \mathbf{o} \rangle, \langle 1, 1 \rangle, \langle \mathbf{o}, 1 \rangle, \langle -1, 1 \rangle]$. Besides the directions ‘east-west’ $E_{\langle 1, 0 \rangle}$ and ‘north-south’ $E_{\langle 0, 1 \rangle}$, this algebra has directions ‘northeast-southwest’ $E_{\langle 1, 1 \rangle}$ and ‘southeast-northwest’ $E_{\langle -1, 1 \rangle}$. There are 17 atoms, namely $D_{\langle 0, 0 \rangle}$ and 16 others, listed counterclockwise from the x -axis: $D_{\langle 1, 0 \rangle}, D_{\langle 1, 0 \rangle}; D_{\langle 1, 1 \rangle},$

	d	\check{d}	e	\check{e}	f	\check{f}
$1'$	d	\check{d}	e	\check{e}	f	\check{f}
a	bcd	$\check{d}\check{e}\check{f}$	bcd	\check{f}	$bcdef$	\check{f}
\check{a}	def	$\check{b}\check{c}\check{d}$	f	$\check{b}\check{c}\check{d}$	f	$\check{b}\check{c}\check{d}\check{e}\check{f}$
b	bcd	$ab\check{d}\check{e}\check{f}$	bcd	$ab\check{f}$	$bcdef$	$ab\check{f}$
\check{b}	$def\check{a}\check{b}$	$\check{b}\check{c}\check{d}$	$f\check{a}\check{b}$	$\check{b}\check{c}\check{d}$	$f\check{a}\check{b}$	$\check{b}\check{c}\check{d}\check{e}\check{f}$
c	d	$ab\check{d}\check{e}\check{f}$	d	$ab\check{f}$	def	$ab\check{f}$
\check{c}	$def\check{a}\check{b}$	\check{d}	$f\check{a}\check{b}$	\check{d}	$f\check{a}\check{b}$	$\check{d}\check{e}\check{f}$
d	d	1	d	$abcd\check{f}$	def	$abcd\check{f}$
\check{d}	1	\check{d}	$f\check{a}\check{b}\check{c}\check{d}$	\check{d}	$f\check{a}\check{b}\check{c}\check{d}$	$\check{d}\check{e}\check{f}$
e	d	$f\check{a}\check{b}\check{c}\check{d}$	e	$1'e\check{e}$	f	$abcd\check{f}$
\check{e}	$abcd\check{f}$	\check{d}	$1'e\check{e}$	\check{e}	$f\check{a}\check{b}\check{c}\check{d}$	\check{f}
f	def	$f\check{a}\check{b}\check{c}\check{d}$	f	$f\check{a}\check{b}\check{c}\check{d}$	f	1
\check{f}	$abcd\check{f}$	$\check{d}\check{e}\check{f}$	$abcd\check{f}$	\check{f}	1	\check{f}

FIGURE 7. Products for $\mathfrak{C}_2[\langle 1, \mathbf{o} \rangle, \langle 1, 1 \rangle, \langle \mathbf{o}, 1 \rangle]$, second part

	$1'$	\hat{a}	\hat{b}	\hat{c}	\hat{d}	\hat{e}	\hat{f}
$1'$	$1'$	\hat{a}	\hat{b}	\hat{c}	\hat{d}	\hat{e}	\hat{f}
\hat{a}	\hat{a}	$1'\hat{a}$	$\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{b}\hat{d}\hat{e}\hat{f}$	$\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{b}\hat{c}\hat{d}\hat{f}$	$\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$
\hat{b}	\hat{b}	$\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	1	$\hat{a}\hat{b}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$
\hat{c}	\hat{c}	$\hat{b}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{d}\hat{e}\hat{f}$	$1'\hat{c}$	$\hat{a}\hat{b}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{d}\hat{f}$	$\hat{a}\hat{b}\hat{d}\hat{e}\hat{f}$
\hat{d}	\hat{d}	$\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{d}\hat{e}\hat{f}$	1	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$
\hat{e}	\hat{e}	$\hat{b}\hat{c}\hat{d}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{f}$	$\hat{a}\hat{b}\hat{d}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{f}$	$1'\hat{e}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{f}$
\hat{f}	\hat{f}	$\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}$	$\hat{a}\hat{b}\hat{c}\hat{d}\hat{f}$	1

FIGURE 8. Products for the symmetric subalgebra of $\mathfrak{C}_2[\langle 1, \mathbf{o} \rangle, \langle 1, 1 \rangle, \langle \mathbf{o}, 1 \rangle]$

$D_{\langle 1,1 \rangle}, D_{\langle 1,1 \rangle}; D_{\langle 0,1 \rangle}, D_{\langle 0,1 \rangle}, D_{\langle 0,1 \rangle}; D_{\langle -1,1 \rangle}, D_{\langle -1,1 \rangle}, D_{\langle -1,1 \rangle}; D_{\langle -1,0 \rangle}, D_{\langle -1,0 \rangle},$
 $D_{\langle -1,0 \rangle}; D_{\langle -1,-1 \rangle}, D_{\langle -1,-1 \rangle}, D_{\langle -1,-1 \rangle}; D_{\langle 0,-1 \rangle}, D_{\langle 0,-1 \rangle}, D_{\langle 0,-1 \rangle}; D_{\langle 1,-1 \rangle},$
 $D_{\langle 1,-1 \rangle}, D_{\langle 1,-1 \rangle}; D_{\langle 1,0 \rangle}.$

The 2-dimensional compass algebra $\mathfrak{C}_2[\langle 1, \mathbf{o} \rangle]$ has just four atoms, namely $D_{\langle 0,0 \rangle}, D_{\langle 1,0 \rangle}, D_{\langle -1,0 \rangle}$, and $F = (\mathbb{R}^2 \times \mathbb{R}^2) \cdot \overline{D_{\langle 0,0 \rangle} + D_{\langle 1,0 \rangle} + D_{\langle -1,0 \rangle}}$. Note that F is a symmetric relation, *i.e.*, $\check{F} = F$, unlike $D_{\langle 1,0 \rangle}$ or $D_{\langle -1,0 \rangle}$. The

	1'	a	\check{a}	b
1'	1'	a	\check{a}	b
a	a	a	$1'a\check{a}$	b
\check{a}	\check{a}	$1'a\check{a}$	\check{a}	b
b	b	b	b	1

FIGURE 9. Products for $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle]$

points of the plane which are in the relation F to the origin are all those which lie in the upper half plane or lower half plane (*i.e.*, not on the x -axis). Let $1' = D_{\langle 0,0 \rangle}$, $a = D_{\langle 1,0 \rangle}$, $\check{a} = D_{\langle -1,0 \rangle}$, and $b = \overline{1' + a + \check{a}}$. Then the cyclesets of $\mathfrak{C}_2[\langle \mathbf{1}, \mathbf{o} \rangle]$ are $[1', 1', 1']$, $[1', a, a]$, $[a, 1', a]$, $[1', b, b]$, $[a, a, a]$, $[a, b, b]$, $[b, b, b]$, and the relative products of atoms are given in Fig.9. This algebra illustrates a general phenomenon. If $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^n$ are pairwise linearly independent but do not span \mathbb{R}^n , then $\mathfrak{C}_n[\mathbf{v}_1, \dots, \mathbf{v}_m]$ will have only one atom for the subspace orthogonal to the subspace spanned by $\mathbf{v}_1, \dots, \mathbf{v}_m$. Notice that this situation must arise whenever the number of directions is less than the number of dimensions, *i.e.*, whenever $m < n$.

Now we consider 3-dimensional compass algebras. Let $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$, $\mathbf{w} = \langle 1, 1, 0 \rangle$, $\mathbf{x} = \langle -1, 1, 0 \rangle$ and $\mathbf{y} = \langle 0, 0, 1 \rangle$. The 3-dimensional compass algebra generated by a single vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}\}$ has 4 atoms. The algebra generated by any two vectors in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}\}$ has 10 atoms. Note that $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ all lie in the same 2-dimensional subspace. Hence any three vectors in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ generate a 3-dimensional compass algebra with 14 atoms, while $\mathfrak{C}_3[\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}]$ has 18 atoms. The vector \mathbf{y} and any two vectors in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ form a linearly independent set, and generate a compass algebra with 27 atoms. The vector \mathbf{y} and any three vectors in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ generate a compass algebra with 39 atoms. Finally, $\mathfrak{C}_3[\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}]$ has 51 atoms.

Not every compass algebra determined by a finite set of vectors is finite. Let $\mathbf{z} = \langle 1, 1, 1 \rangle$. Then $\mathfrak{C}_3[\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}] = \mathfrak{C}_3[\langle \mathbf{1}, \mathbf{o}, \mathbf{o} \rangle, \langle \mathbf{o}, \mathbf{1}, \mathbf{o} \rangle, \langle \mathbf{o}, \mathbf{o}, \mathbf{1} \rangle, \langle \mathbf{1}, \mathbf{1}, \mathbf{1} \rangle]$ is infinite. To see this, let $X_0 = E_{\mathbf{u}}$, $Y_0 = E_{\mathbf{v}}$, $Z_0 = E_{\mathbf{y}}$, and, for every integer n , $X_{n+1} = X_n; E_{\mathbf{z}} \cdot Y_n; Z_n$, $Y_{n+1} = Y_n; E_{\mathbf{z}} \cdot X_n; Z_n$, and $Z_{n+1} = Z_n; E_{\mathbf{z}} \cdot X_n; Y_n$. Then X_n , Y_n , and Z_n are all distinct equivalence relations for every n . In

particular,

$$\begin{array}{lll}
X_0 = E_{\langle 1,0,0 \rangle} & Y_0 = E_{\langle 0,1,0 \rangle} & Z_0 = E_{\langle 0,0,1 \rangle} \\
X_1 = E_{\langle 0,1,1 \rangle} & Y_1 = E_{\langle 1,0,1 \rangle} & Z_1 = E_{\langle 1,1,0 \rangle} \\
X_2 = E_{\langle 2,1,1 \rangle} & Y_2 = E_{\langle 1,2,1 \rangle} & Z_2 = E_{\langle 1,1,2 \rangle} \\
X_3 = E_{\langle 2,3,3 \rangle} & Y_3 = E_{\langle 3,2,3 \rangle} & Z_3 = E_{\langle 3,3,2 \rangle} \\
X_4 = E_{\langle 6,5,5 \rangle} & Y_4 = E_{\langle 5,6,5 \rangle} & Z_4 = E_{\langle 5,5,6 \rangle} \\
\vdots & \vdots & \vdots
\end{array}$$

4. Some NP-complete Constraint Satisfiability Problems

Let \mathfrak{A} be a relation algebra. An \mathfrak{A} -matrix is a matrix of elements of \mathfrak{A} . Suppose M is an n -by- n \mathfrak{A} -matrix. We say M is *zeroless* if no entry in M is 0, and M is *closed* if $M_{ii} \leq 1$, $(M_{ij})^\circ = M_{ji}$, and $M_{ij} \leq M_{ik}; M_{kj}$ whenever $1 \leq i, j, k \leq n$. If N is another n -by- n matrix, we say N is a *reduction* of M , in symbols, $N \leq M$, if $N_{ij} \leq M_{ij}$ whenever $1 \leq i, j \leq n$. If X is a set of elements of \mathfrak{A} , we say M is *bounded by* X if every entry of M is included in some element of X .

A *binary constraint matrix* is a matrix of binary relations. An n -by- n binary constraint matrix M determines an n -ary relation $R(M) = \{\langle p_1, \dots, p_n \rangle : \langle p_i, p_j \rangle \in M_{ij} \text{ whenever } 1 \leq i, j \leq n\}$. The matrix M specifies a *binary constraint problem*. The *solutions* to this problem are the n -tuples in $R(M)$, and the problem is *solvable* if it has a solution. Let U be the set of elements that appear in any pair in any relation in M . Each n -tuple $\langle p_1, \dots, p_n \rangle$ of elements of U corresponds naturally to an n -by- n matrix N of atoms of $\mathfrak{Re}(\mathfrak{U})$, where $N_{ij} = \{\langle p_i, p_j \rangle\}$ whenever $1 \leq i, j \leq n$. Note that $\langle p_1, \dots, p_n \rangle$ is a solution to M if and only if its corresponding matrix N is a reduction of M . Furthermore, as a binary constraint problem, M has a solution just in case there is a closed zeroless reduction of M bounded by the set of atoms of $\mathfrak{Re}(\mathfrak{U})$.

This last observation permits us to generalize the concept of constraint satisfaction to arbitrary atomic relation algebras. Let \mathfrak{A} be an atomic relation algebra and let M be an \mathfrak{A} -matrix. We say that M is *proto-solvable over* \mathfrak{A} if there is a closed zeroless reduction of M which is bounded by the set of atoms of \mathfrak{A} . Note that if N is a closed zeroless \mathfrak{A} -matrix bounded by the atoms of \mathfrak{A} , then all the entries in N must actually be atoms of \mathfrak{A} . Such a matrix, whose entries are all atoms of \mathfrak{A} , is called *atomic*. So the \mathfrak{A} -matrix M is proto-solvable if it has a closed atomic reduction N . Such an N is called a *proto-solution*. The *constraint satisfiability problem* for an atomic relation algebra \mathfrak{A} is this: given an \mathfrak{A} -matrix, determine whether it has a proto-solution.

For an $\mathfrak{Re}(\mathfrak{U})$ -matrix M , the solutions and proto-solutions (over $\mathfrak{Re}(\mathfrak{U})$) are in a one-to-one correspondence, as observed above. But for matrices over atomic subalgebras of $\mathfrak{Re}(\mathfrak{U})$, such a correspondence may not exist. Indeed, it is easy to find a set U , a finite subalgebra \mathfrak{A} of $\mathfrak{Re}(\mathfrak{U})$, and an \mathfrak{A} -matrix M such

that M has a proto-solution but no solution. For example, let $U = \{1, 2, 3\}$, let \mathfrak{A} be the subalgebra of $\mathfrak{Re}(\mathfrak{U})$ with atoms $1'$ and $0'$ (\mathfrak{A} is isomorphic to $\mathfrak{C}_1[\langle \circ \rangle]$),

and let $M = \begin{pmatrix} 1' & 0' & 0' & 0' \\ 0' & 1' & 0' & 0' \\ 0' & 0' & 1' & 0' \\ 0' & 0' & 0' & 1' \end{pmatrix}$. Then M is a proto-solution of itself, but

it has no solution, since any solution of M must be a quadruple $\langle p_1, p_2, p_3, p_4 \rangle$ with distinct entries, but there are only three elements in U . On the other hand, M can be considered as a $\mathfrak{C}_1[\langle \circ \rangle]$ -matrix, in which case it does have solutions, namely all quadruples of distinct real numbers.

We have seen that proto-solutions can exist when solutions do not. It is also possible for solutions to exist when proto-solutions do not: an infinite atomic subalgebra \mathfrak{A} of $\mathfrak{Re}(\mathfrak{U})$, where U is a countable infinite set, and an \mathfrak{A} -matrix M with a solution but no proto-solution over \mathfrak{A} . Examples of this are more difficult to construct, but can be found in [18] and [19]. For such an example, however, it is necessary that \mathfrak{A} be infinite [16, Theorem 5.7].

For the IA, the situation is quite nice. An IA-matrix has a solution if and only if it has a proto-solution [17] [16] and this is true for all isomorphic copies of the IA which are embedded in algebras $\mathfrak{Re}(\mathfrak{U})$ where U is not necessarily the set of events based on real numbers. Constraint satisfiability for the IA is NP-complete. A sketch of a proof of this was given in [43]. The idea of that proof is to reduce the 3-clause satisfiability problem (for propositional calculus) to constraint satisfiability for the IA. (Additional details for that proof are given in [44].) Another proof is sketched in [40], where graph-colorability is reduced to constraint satisfiability for the IA. Both of these proofs deal with solutions, not proto-solutions, but, in view of the remarks made above, this makes no difference to the IA.

The NP-completeness of the constraint satisfiability problem for the IA follows from Theorem 2 below. This theorem is not restricted to the IA and indeed applies some compass algebras. It also applies to infinite algebras, such as the Allen-Hayes algebra, and to nonrepresentable algebras.

Theorem 2. *Assume \mathfrak{A} is a relation algebra with elements $x, y, z \neq 0$, such that*

- (i) $1', x, \check{x}, y, \check{y}, z, \check{z}$ are pairwise disjoint,
- (ii) $z \cdot z; y = 0$,
- (iii) $y \cdot x; y = 0$,
- (iv) $x \cdot x; y = 0$,
- (v) $z \cdot x; z = 0$,
- (vi) $y \cdot y; z = 0$,
- (vii) $x \cdot z; x = 0$,
- (viii) $z \leq x; y, x \leq z; \check{y}, y \leq \check{x}; z$,
- (ix) $1' \leq x; \check{x} \cdot \check{x}; x \cdot y; \check{y} \cdot \check{y}; y \cdot z; \check{z} \cdot \check{z}; z$.

Then following problem is NP-complete: **(R)** Determine whether a matrix M over \mathfrak{A} has a closed zeroless reduction bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$.

Proof. It suffices to show that Graph 3-Colorability [9] is reducible to **(R)**. Let $G = \langle V, E \rangle$ be a graph (*i.e.*, E is a symmetric binary relation on V) that is disjoint from the identity relation on V). We may assume without loss of generality that the set V of vertices of G is $\{4, \dots, |V| + 3\}$, where $|V|$ is the cardinality of V . Let $n = |V| + 3$. Let M be the n -by- n \mathfrak{A} -matrix determined by the following stipulations:

- (i) $M_{ii} = 1'$ for $1 \leq i \leq n$,
- (ii) $M_{12} = x$, $M_{21} = \check{x}$, $M_{23} = y$, $M_{32} = \check{y}$, $M_{13} = z$, $M_{31} = \check{z}$,
- (iii) $M_{i1} = 1' + \check{x} + \check{z}$, $M_{1i} = 1' + x + z$, $M_{i2} = 1' + x + \check{y}$, $M_{2i} = 1' + \check{x} + y$,
 $M_{i3} = 1' + y + z$, and $M_{3i} = 1' + \check{y} + \check{z}$ whenever $i \in V$ (*i.e.*, $4 \leq i \leq n$),
- (iv) $M_{ij} = M_{ji} = x + \check{x} + y + \check{y} + z + \check{z}$ whenever $i, j \in V$ and $\langle i, j \rangle \in E$,
- (v) in all other cases, $M_{ij} = 1$.

We will show that there is a natural one-to-one correspondence between 3-colorings of the graph G and closed zeroless reductions of M which are bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$. It follows that M has a closed zeroless reduction bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$ just in case the graph G is 3-colorable.

Suppose that N is a closed zeroless reduction of M which is bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$. We will show that N determines a 3-coloring $\gamma : V \rightarrow \{1, 2, 3\}$ of G . First, since $1', x, \check{x}, y, \check{y}, z, \check{z}$ are pairwise disjoint, N is zeroless, and N is bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$, we concluded that if $1 \leq i, j \leq n$, then exactly one of the following seven statements holds: $N_{ij} \leq 1'$, $N_{ij} \leq x$, $N_{ij} \leq \check{x}$, $N_{ij} \leq y$, $N_{ij} \leq \check{y}$, $N_{ij} \leq z$, $N_{ij} \leq \check{z}$. Now we look at the possible values of N_{i1} , N_{i2} , and N_{i3} for an arbitrary $i \in V$, *i.e.*, for $4 \leq i \leq n$. Since $N \leq M$, we have $N_{12} \leq x$, $N_{23} \leq y$, $N_{13} \leq z$, $N_{i1} \leq 1' + \check{x} + \check{z}$, $N_{i2} \leq 1' + x + \check{y}$, and $N_{i3} \leq 1' + y + z$. If $N_{i1} \leq 1'$, then $N_{i2} \leq N_{i1}; N_{12} \leq 1'; x = x$ and $N_{i3} \leq N_{i1}; N_{13} \leq 1'; z = z$. Similarly, if $N_{i2} \leq 1'$, then $N_{i1} \leq N_{i2}; N_{21} \leq 1'; \check{x} = \check{x}$ and $N_{i3} \leq N_{i2}; N_{23} \leq 1'; y = y$. Finally, if $N_{i3} \leq 1'$, then $N_{i1} \leq N_{i3}; N_{31} \leq 1'; \check{z} = \check{z}$ and $N_{i2} \leq N_{i3}; N_{32} \leq 1'; \check{y} = \check{y}$. From these observations it follows that $N_{ik} \leq 1'$ for at most one $k \in \{1, 2, 3\}$. To show $N_{ik} \leq 1'$ for at least one $k \in \{1, 2, 3\}$, we assume $N_{i1} \leq \check{x} + \check{z}$, $N_{i2} \leq x + \check{y}$, and $N_{i3} \leq y + z$, and derive a contradiction. There are two cases. First, if $N_{i3} \leq y$, then $N_{i1} \leq (\check{x} + \check{z}) \cdot N_{i3}; N_{31} \leq (\check{x} + \check{z}) \cdot y; \check{z} \leq \check{x}$ and $N_{i2} \leq (x + \check{y}) \cdot N_{i3}; N_{32} \leq (x + \check{y}) \cdot y; \check{y} \leq \check{y}$ by (ii) and (iii), respectively. From these last two equations we get $N_{i2} \leq \check{y} \cdot N_{i1}; N_{12} \leq \check{y} \cdot \check{x}; x = 0$ by (iv), contradicting the assumption that N is zeroless. Second, if $N_{i3} \leq z$, then $N_{i1} \leq (\check{x} + \check{z}) \cdot N_{i3}; N_{31} \leq (\check{x} + \check{z}) \cdot z; \check{z} \leq \check{z}$ and $N_{i2} \leq (x + \check{y}) \cdot N_{i3}; N_{32} \leq (x + \check{y}) \cdot z; \check{y} \leq x$ by (v) and (vi), respectively. From these last two equations we get $N_{i2} \leq x \cdot N_{i1}; N_{12} \leq x \cdot \check{z}; x = 0$ by (vii), again contradicting the assumption that N is zeroless. This exhausts the possibilities. Thus we have $N_{ik} \leq 1'$ for exactly one $k \in \{1, 2, 3\}$. This allows us to define $\gamma : V \rightarrow \{1, 2, 3\}$ by $\gamma(i) = k$ if and only if $N_{ik} \leq 1'$, for every $i \in V$. Now if $\langle i, j \rangle \in E$, then we must have $\gamma(i) \neq \gamma(j)$, for if $\gamma(i) = \gamma(j) = k$, then we have $N_{ik} \leq 1'$ and $N_{jk} \leq 1'$, from which we obtain

$x + \check{x} + y + \check{y} + z + \check{z} = N_{ij} \leq N_{ik}; N_{kj} \leq 1'; \check{1}' = 1'$, contradicting (i). Thus γ is a 3-coloring of G .

For the other direction, if we have a 3-coloring $\gamma : V \rightarrow \{1, 2, 3\}$ of G , we can get a closed zeroless reduction $N \leq M$ which is bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$ as follows. Set $N_{12} = x$, $N_{21} = \check{x}$, $N_{23} = y$, $N_{32} = \check{y}$, $N_{13} = z$, $N_{31} = \check{z}$, and $N_{ii} = 1'$ whenever $1 \leq i \leq n$. For all $i, j \in V$, and every $k \in \{1, 2, 3\}$, set $N_{ik} = N_{\gamma(i)k}$, $N_{ki} = N_{k\gamma(i)}$, and $N_{ij} = N_{\gamma(i)\gamma(j)}$. It follows from (viii) and (ix) that this definition gives a closed zeroless matrix N . Obviously, N is bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$. The fact that γ is a 3-coloring of G is used to show that $N \leq M$. \square

Corollary 3. *Constraint satisfiability is NP-complete for both the algebra $C_2[\langle 1, 0 \rangle, \langle 1, 1 \rangle, \langle 0, 1 \rangle]$ and its symmetric subalgebra.*

Proof. Use Theorem 2 with $x = \hat{a}$, $y = \hat{c}$, and $z = \hat{e}$. \square

Corollary 4. *Constraint satisfiability is NP-complete for both the interval algebra and the Allen-Hayes algebra.*

Proof. Use Theorem 2 with $x = m$, $y = f$, and $z = s$. \square

Theorem 2 applies to a 3-directional compass algebra. For 2-directional compass algebras we need another theorem.

Theorem 5. *Let \mathfrak{A} be a relation algebra with nonzero elements x, y, z such that*

- (i) $1', x, \check{x}, y, \check{y}, z, \check{z}$ are pairwise disjoint,
- (ii) $y \cdot x; x = 0$,
- (iii) $y \cdot x; y = 0$,
- (iv) $y \cdot y; x = 0$,
- (v) $y \cdot y; z = 0$,
- (vi) $y \cdot z; y = 0$,
- (vii) $y \cdot z; z = 0$,
- (viii) $x \cdot z; x = 0$,
- (ix) $x \cdot x; z = 0$,
- (x) $z \leq z; z, z \leq z; \check{z}, z \leq \check{z}; z$,
- (xi) $y \leq y; y, y \leq y; \check{y}, y \leq \check{y}; y$,
- (xii) $y \leq x; z, x \leq y; \check{z}, z \leq \check{x}; y$,
- (xiii) $y \leq z; x, z \leq y; \check{x}, x \leq \check{z}; y$,
- (xiv) $z \leq z; y, z \leq z; \check{y}, y \leq \check{z}; z$,
- (xv) $z \leq y; z, y \leq z; \check{z}, z \leq \check{y}; z$,
- (xvi) $z \leq z; x, z \leq z; \check{x}, x \leq \check{z}; z$,
- (xvii) $z \leq x; z, x \leq z; \check{z}, z \leq \check{x}; z$,
- (xviii) $1' \leq x; \check{x} \cdot \check{x}; x \cdot y; \check{y} \cdot \check{y}; y \cdot z; \check{z} \cdot \check{z}; z$.

Then the following problem is NP-complete: (R) Determine whether a network N over \mathfrak{A} with labels in $\{y, x + y + z, z + \check{z}\}$ has a closed zeroless reduction bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$.

Proof. As in the previous proof, we show that Graph 3-Colorability [9] is reducible to **(R)**. Let $G = \langle V, E \rangle$ be a graph with vertex set $V = \{3, \dots, |V|+2\}$, Let $n = |V| + 2$. Let M be the n -by- n \mathfrak{A} -matrix determined by the following stipulations: $M_{12} = y$, $M_{21} = \check{y}$, $M_{1i} = M_{i2} = x + y + z$ for every $i \in V$, $M_{ij} = z + \check{z}$ whenever $i, j \in V$ and $\langle i, j \rangle \in E$, The 3-colorings of G correspond to closed zeroless reductions of M which are bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$.

Suppose that N is a closed zeroless reduction of M which is bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$. We show that N determines a 3-coloring $\gamma : V \rightarrow \{1, 2, 3\}$. First, if $1 \leq i, j \leq n$, then exactly one of the following seven statements holds: $N_{ij} \leq 1'$, $N_{ij} \leq x$, $N_{ij} \leq \check{x}$, $N_{ij} \leq y$, $N_{ij} \leq \check{y}$, $N_{ij} \leq z$, and $N_{ij} \leq \check{z}$. Now we look at the possible values of N_{1i} and N_{i2} for an arbitrary $i \in V$. We have $N_{12} \leq y$, $N_{1i} \leq x+y+z$, $N_{i2} \leq x+y+z$. Hence there are nine cases, six of which are ruled out because they contradict one of the hypotheses. For example, if $N_{1i} \leq x$ and $N_{i2} \leq y$, then by (iii) we have $N_{12} \leq y \cdot N_{1i}; N_{i2} \leq y \cdot x; y = 0$, contradicting the assumption that N is zeroless. The following table shows which cases are ruled out by hypotheses (ii)–(vii).

	$N_{i2} \leq x$	$N_{i2} \leq y$	$N_{i2} \leq z$
$N_{1i} \leq x$	No, by (ii).	No, by (iii).	
$N_{1i} \leq y$	No, by (iv).		No, by (v).
$N_{1i} \leq z$		No, by (vi).	No, by (vii).

The remaining three cases are used to define $\gamma : V \rightarrow \{1, 2, 3\}$. For every $i \in V$,

$$\gamma(i) = \begin{cases} 1 & \text{if } N_{1i} \leq z \text{ and } N_{i2} \leq x \\ 2 & \text{if } N_{1i} \leq y \text{ and } N_{i2} \leq y \\ 3 & \text{if } N_{1i} \leq x \text{ and } N_{i2} \leq z \end{cases}$$

Now we must show $\gamma(i) \neq \gamma(j)$ whenever $\langle i, j \rangle \in E$. Since N is closed and $N \leq M$, we have $N_{ij} \leq z + \check{z}$ and $N_{ji} \leq z + \check{z}$. If $\gamma(i) = 1$ then $N_{i2} \leq x$, so by (ix) we get $N_{j2} \leq (x + y + z) \cdot N_{ji}; N_{i2} \leq (x + y + z) \cdot (z + \check{z}); x \leq (x + y + z) \cdot (z; x + \check{z}; x) \leq y + z$. Therefore, either $N_{j2} \leq y$ and $\gamma(j) = 2$, or else $N_{j2} \leq z$ and $\gamma(j) = 3$. Thus $\gamma(i) \neq \gamma(j)$. If $\gamma(i) = 2$ then $N_{i2} \leq y$, so $N_{j2} \leq (x+y+z) \cdot N_{ji}; N_{i2} \leq (x+y+z) \cdot (z+\check{z}); y \leq (x+y+z) \cdot (z; y+\check{z}; y) \leq x+z$. by (vi). Thus either $N_{j2} \leq x$ and $\gamma(j) = 1$, or else $N_{j2} \leq z$ and $\gamma(j) = 3$. Again, $\gamma(i) \neq \gamma(j)$. Finally, if $\gamma(i) = 3$ then $N_{1i} \leq x$, so $N_{1j} \leq (x + y + z) \cdot N_{1i}; N_{ij} \leq (x + y + z) \cdot x; (z + \check{z}) \leq (x + y + z) \cdot (x; z + x; \check{z}) \leq y + z$ by (viii). Either $N_{j2} \leq x$ and $\gamma(j) = 2$, or else $N_{j2} \leq z$ and $\gamma(j) = 1$. Hence $\gamma(i) \neq \gamma(j)$. This completes the proof that $\gamma(i) \neq \gamma(j)$ whenever $\langle i, j \rangle \in E$, and shows that γ is a 3-coloring of G .

For the other direction, if we have a 3-coloring $\gamma : V \rightarrow \{1, 2, 3\}$, we can get a closed zeroless reduction $N \leq M$ which is bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$. Set

$N_{12} = y$, $N_{21} = \check{y}$, and $N_{ii} = 1'$ whenever $1 \leq i \leq n$. For all $i, j \in V$, set

$$N_{ij} = \begin{cases} 1' & \text{if } \gamma(i) = \gamma(j) \\ z & \text{if } \gamma(i) > \gamma(j) , \\ \check{z} & \text{if } \gamma(i) < \gamma(j) \end{cases}$$

$$N_{1i} = \begin{cases} z & \text{if } \gamma(i) = 1 \\ y & \text{if } \gamma(i) = 2 , \\ x & \text{if } \gamma(i) = 3 \end{cases} , \quad N_{i2} = \begin{cases} x & \text{if } \gamma(i) = 1 \\ y & \text{if } \gamma(i) = 2 , \\ z & \text{if } \gamma(i) = 3 \end{cases}$$

$$N_{i1} = \begin{cases} \check{z} & \text{if } \gamma(i) = 1 \\ \check{y} & \text{if } \gamma(i) = 2 , \\ \check{x} & \text{if } \gamma(i) = 3 \end{cases} , \quad N_{2i} = \begin{cases} \check{x} & \text{if } \gamma(i) = 1 \\ \check{y} & \text{if } \gamma(i) = 2 . \\ \check{z} & \text{if } \gamma(i) = 3 \end{cases}$$

It follows from (x)–(xviii) that N is closed and zeroless. Obviously, N is bounded by $\{1', x, \check{x}, y, \check{y}, z, \check{z}\}$. The fact that γ is a 3-coloring of G is used to show that $N \leq M$. \square

To see the necessity of (x)–(xviii), consider the following example. Let $G = \langle V, E \rangle$ where $V = \{3, 4, 5\}$ and $E = \emptyset$. Let $\gamma : V \rightarrow \{1, 2, 3\}$ be the 3-coloring of G defined by $\gamma(i) = i - 2$ for every $i \in V$. The resulting N is shown below. The matrix N is closed if and only if (x)–(xviii) hold.

$$N = \begin{pmatrix} 1' & y & z & y & x \\ \check{y} & 1' & \check{x} & \check{y} & \check{z} \\ \check{z} & x & 1' & \check{z} & \check{z} \\ \check{y} & y & z & 1' & \check{z} \\ \check{x} & z & z & z & 1' \end{pmatrix}$$

Corollary 6. *Constraint satisfiability for $C_2[\langle 1, 0 \rangle, \langle 0, 1 \rangle]$ is NP-complete. The same is true for any compass algebra with at least two directions.*

Proof. By Theorem 5, with $x = a$, $y = c$, and $z = d$. \square

These theorems can be extended to show that essentially all but the most trivial compass and interval algebras have NP-hard constraint satisfaction problems.

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