1. Axioms and rules of relevance logic

axioms of **R**:

(1)	$A \to A$	Self-Implication
(2)	$A \wedge B \to A$	Conjunction Elim.
(3)	$A \wedge B \to B$	Conjunction Elim.
(4)	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	Conjunction Intro.
(5)	$A \to A \vee B$	Disjunction Intro.
(6)	$B \to A \vee B$	Disjunction Intro.
(7)	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	Disjunction Elim.
(8)	$A \wedge (B \vee C) \to (A \wedge B) \vee C$	Distribution
(9)	$\sim\sim A \to A$	Double Negation
(10)	$((A \to A) \to B) \to B$	Specialized Assertion
(11)	$(A \to B) \to ((C \to A) \to (C \to B))$	Prefixing
(12)	$(A \to B) \to ((B \to C) \to (A \to C))$	Suffixing^{c}
(13)	$A \to ((A \to B) \to B)$	Assertion^{c}
(14)	$(A \to \sim B) \to (B \to \sim A)$	Contraposition^{c}
(15)	$(A \to (B \to C)) \to (B \to (A \to C))$	Permutation^{c}
(16)	$A \to ((A \to A) \to A)$	Demodalizer^{c}
(17)	$(A \to (A \to B)) \to (A \to B)$	Contraction^{d}
(18)	$(A \to \sim A) \to \sim A$	Reductio^{d}
(19)	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	Self-Distribution^{cd}

additional axioms of **RM**:

(20)	$A \to (A \to A)$	Mingle^{t}
(21)	$(A \to B) \to (A \to (A \to B))$	Expansion^{t}

rules of both **R** and **RM**:

(22)	$A, A \to B \vdash B$	*Modus Ponens*
(23)	$A, B \vdash A \wedge B$	Adjunction

2. Formulas as sets

Df. Given a set U and a family of subsets $\mathcal{S} \subseteq Sb(U)$ closed under these operations:

$$A \wedge B := A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$A \vee B := A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$\sim A := \{x : x \in U \text{ and } x \notin A\}$$

$$A \to B := \{x : \forall x(\text{if } x \in A \text{ then } x \in B)\} = \sim A \vee B$$

$$(\, A \subseteq B \quad \text{iff} \quad U \subseteq A \to B \,)$$

Let $A = A(p, q, \dots)$ be a formula with variables p, q, ... Define

$$\mathcal{S} \models A \quad \text{iff} \quad U \subseteq A \text{ for all } p, q, \dots \in \mathcal{S} \quad (\text{“}\mathcal{S} \text{ models } A\text{”})$$

Th. If $\mathcal{S} \models A$ and $\mathcal{S} \models A \to B$ then $\mathcal{S} \models B$.

Th. If $\mathcal{S} \models A$ and $\mathcal{S} \models B$ then $\mathcal{S} \models A \wedge B$.

Th. If A is one of (1)–(21) then $\mathcal{S} \models A$ for every family of sets $\mathcal{S}$.

Th. If $\mathbf{RM} \vdash A$ then every family of sets models A.

Th. Every family of sets models $A \to (B \to A)$, but $\mathbf{RM} \nvdash A \to (B \to A)$.

3. Formulas as relations

Df. Given a set U and a family of relations $\mathcal{R} \subseteq Sb(U^2)$ closed under these operations:

$$A \wedge B := A \cap B = \{\langle x, y\rangle : \langle x, y\rangle \in A \text{ and } \langle x, y\rangle \in B\}$$

$$A \vee B := A \cup B = \{\langle x, y\rangle : \langle x, y\rangle \in A \text{ or } \langle x, y\rangle \in B\}$$

$$\sim A := \{\langle x, y\rangle : x, y \in U \text{ and } \langle y, x\rangle \notin A\}$$

$$A \to B := \{\langle x, y\rangle : \forall z(\text{if } \langle z, x\rangle \in A \text{ then } \langle z, y\rangle \in B)\}$$

$$(e.g., \text{“}\in \to \in = \subseteq\text{”})$$

Let $A = A(p, q, \ldots)$ be a formula with variables p, q, $\ldots$. Define

$$\mathcal{R} \models A \quad \text{iff} \quad \mathsf{Id} \subseteq A \text{ for all } p, q, \dots \in \mathcal{R} \quad (\text{“}\mathcal{R} \text{ models } A\text{”})$$

where $\mathsf{Id} := \{\langle x, x\rangle : x \in U\}$.

Th. $A \subseteq B$ iff $\mathsf{Id} \subseteq A \to B$.

Th. If $\mathcal{R} \models A$ and $\mathcal{R} \models A \to B$ then $\mathcal{R} \models B$.
Th. If $\mathcal{R} \models A$ and $\mathcal{R} \models B$ then $\mathcal{R} \models A \wedge B$.
Th. If A is one of (1)–(11) then $\mathcal{R} \models A$ for every family of relations $\mathcal{R}$.

4. Calculus of relations

Df. $A|B := \{\langle x,z\rangle : \exists y(\langle x,y\rangle \in A \text{ and } \langle y,z\rangle \in B)\}$.

Df. $A \dagger B := \{\langle x,z\rangle : \forall y(\langle x,y\rangle \in A \text{ or } \langle y,z\rangle \in B))\}$.

Df. $A^{-1} := \{\langle y,x\rangle : \langle x,y\rangle \in A\}$.

Th. $\sim\! A = \overline{A^{-1}}$.

Th. $A \to B = \overline{A^{-1}} \dagger B = \overline{A^{-1}|\overline{B}}$.

Th. $A \to (B \to C) = (B|A) \to C$.

Th. $A|(A \to B) \subseteq B$.

Th. If $A \subseteq B$ then $B \to X \subseteq A \to X$.

Th. If $A \subseteq B$ then $X \to A \subseteq X \to B$.

Df. $\mathcal{R}$ is **commuting** iff $A|B = B|A$ for all $A, B \in \mathcal{R}$.

Th. If A is one of (12)–(16) then every commuting family of relations models A.

Df. $\mathcal{R}$ is **dense** iff $A \subseteq A|A$ for every $A \in \mathcal{R}$.

Th. If A is one of (17)–(18) then every dense family of relations models A.

Th. If A is (19) then every commuting dense family of relations models A.

Df. $\mathcal{R}$ is **transitive** iff $A|A \subseteq A$ for every $A \in \mathcal{R}$.

Th. If A is one of (20)–(21) then every transitive family of relations models A.

Th. If $\mathbf{R} \vdash A$ then $\mathcal{R} \models A$ for every commuting dense family of relations $\mathcal{R}$.

Th. If $\mathbf{RM} \vdash A$ then $\mathcal{R} \models A$ for every commuting dense transitive family of relations $\mathcal{R}$.

Th. There are families of relations that are commuting and dense but not transitive so $\mathbf{R} \nvdash$ Mingle, *etc.*

? If every commuting dense family of relations models A, does $\mathbf{R} \vdash A$?

? If every commuting dense transitive family of relations models A, does $\mathbf{RM} \vdash A$?

5. Example

$U = \mathbb{Q}$ is the set of rational numbers,

$$L := \{\langle x, y\rangle : x < y\},$$
$$G := \{\langle x, y\rangle : x > y\},$$
$$\mathsf{Id} := \{\langle x, x\rangle : x \in U\},$$
$$\mathcal{R} := \{\emptyset,\ \mathsf{Id},\ \mathsf{Id} \cup L,\ U^2,\ \mathsf{Id} \cup G,\ L,\ L \cup G,\ G\}.$$

$\mathcal{R}$ is dense and commuting.

$\mathcal{R}$ is not transitive ($L \cup G$ is not a transitive relation), so $\mathbf{R} \not\vdash$ Mingle.

Nine nonempty subsets of $\mathcal{R}$ are closed under $\rightarrow$, $\sim$, $\wedge$, $\vee$, *e.g.*,

$\mathcal{R}_0 := \{L, \mathsf{Id} \cup L\} = \{<, \leq\}$,

$\mathcal{R}_1 := \{G, \mathsf{Id} \cup G\} = \{>, \geq\}$.

$\mathcal{R}_0$ and $\mathcal{R}_1$ are transitive.

$X \rightarrow Y = \emptyset$ for all $X \in \mathcal{R}_0$ and $Y \in \mathcal{R}_1$.

If $A = A(p, q, \dots)$ and $p, q, \dots \in \mathcal{R}_0$ then $A \in \mathcal{R}_0$, similarly for $\mathcal{R}_1$.

Suppose Suppose $A = A(p, q, \dots)$ and $B = B(r, s, \dots)$ share no variable.

Assign variables so that $p, q, \dots \in \mathcal{R}_0$ and $r, s, \dots \in \mathcal{R}_1$.

Then $A \in \mathcal{R}_0$ and $B \in \mathcal{R}_1$, so $A \rightarrow B = \emptyset \not\supseteq \mathsf{Id}$, so $\mathcal{R} \not\models A \rightarrow B$.

Therefore $A \rightarrow B$ is not a theorem of $\mathbf{R}$.

This proof is from Routley-Meyer 1973, *The semantics of entailment. I,* using a particular normal relevant model structure $\mathcal{M}$ instead of $\mathcal{R}$.

$\mathcal{M}$ is the atom structure of the integral representable relation algebra $\mathsf{1}_3$.

$\mathcal{R}$ is the set of relations in a particular representation of $\mathsf{1}_3$.

$\mathsf{1}_3$ has no representation on a finite set.

Is there a *finitely* representable RA which can be used for this proof?

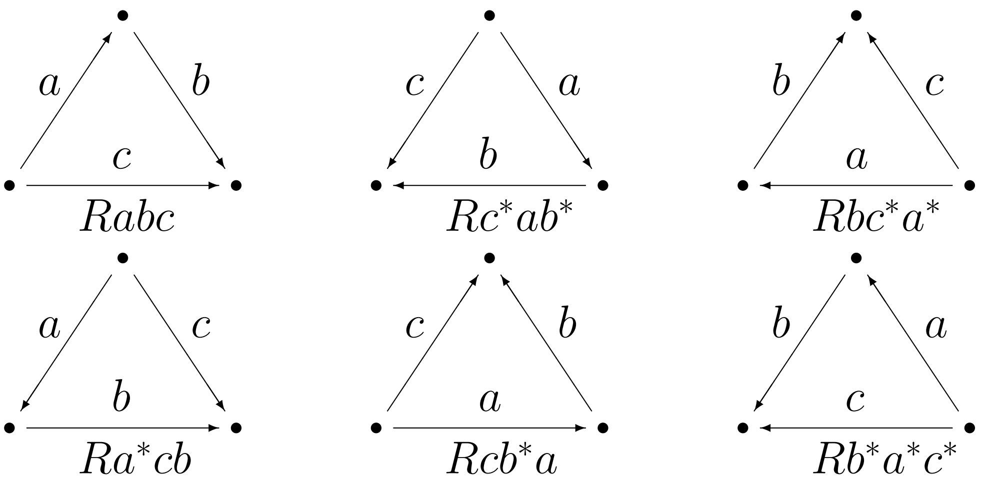
6. Relevant model structures and relation algebras

Let $\mathcal{M} = \langle 0, K, R, {}^* \rangle$ where $0 \in K$, $R \subseteq K^3$, ${}^* : K \to K$. Definitions:

Th. R^2abcd iff $\exists x \in K(Rabx, Rxcd)$
Th. $R^2a(bc)d$ iff $\exists x \in K(Rbcx, Raxd)$

Conditions on $\mathcal{M}$ (for all $a, b, c, d \in K$):

(1) $R0ab$ iff $a = b$ (identity)
(2) $Raaa$ (density)
(3) $R^2abcd \implies R^2a(bc)d$ (associativity, Pasch)
(4) $Rabc \implies Rc^*ab^*$ (rotation)
(5) $Rabc \implies Ra^*cb$ (reflection)
(6) $a^{**} = a$ (involution)
(7) $Rabc \implies Rbac$ (commutativity)



Df. $\mathcal{M}$ is a **normal relevant model structure** iff
$\mathcal{M} \models$(1), (2), (3), (4), (6), (7).
Th. $\mathcal{M}$ is the atom structure of an integral relation algebra $\mathfrak{A}$ iff
$\mathcal{M} \models$(1), (3), (4), (5), (6).
Th. $\mathcal{M}$ is the atom structure of a dense integral relation algebra $\mathfrak{A}$ iff
$\mathcal{M} \models$(1), (2), (3), (4), (5), (6).
Th. $\mathcal{M}$ is the atom structure of a commutative integral relation algebra $\mathfrak{A}$ iff
$\mathcal{M} \models$(1), (3), (4), (5), (6), (7).

7. Ternary relational semantics

Th. Given $\mathcal{M} = \langle 0, K, R, ^* \rangle$ where $0 \in K$, $R \subseteq K^3$, $^* : K \to K$, every valuation ν : variables $\times K \to \{T, F\}$, determines an interpretation I : formulas $\times K \to \{T, F\}$ by

$I(p, a) = \nu(p, a)$,

$I(A \wedge B, a) = T$ iff $I(A, a) = T$ and $I(B, a) = T$,

$I(A \vee B, a) = T$ iff $I(A, a) = T$ or $I(B, a) = T$,

$I(A \to B, a) = T$ iff $\forall b, c \in K$(if $Rabc, I(A, b) = T$ then $I(B, c) = T$),

$I({\sim}A, a) = T$ iff $I(A, a^*) = F$.

Df. A is **true on** valuation ν (or interpretation I) at $a \in K$, if $I(A, a) = T$.

Df. A is **verified on** ν (or I) if $I(A, 0) = T$.

Df. A is **valid** in $\mathcal{M}$ if A is verified on every valuation ν.

Th. (Routley-Meyer) A is valid in every normal relevant model structure iff $\mathbf{R} \vdash A$.

? Is A is valid in every normal relevant model structure satisfying (5) (reflection) iff $\mathbf{R} \vdash A$?

? Is A is valid in every atom structure of a commutative dense integral relation algebra iff $\mathbf{R} \vdash A$?