The Bean Counter:

a model of self-organized criticality

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based on

How Nature Works

the science of self-organized criticality

Per Bak, Springer-Verlag 1996

p. xi "Self-organized criticality is a new way of viewing nature. The basic picture is one where nature is perpetually out of balance, but organized in a poised state—the critical state—where anything can happen within well-defined statistical laws. The aim of the science of self-organized criticality is to yield insight into the fundamental question of why nature is complex, not simple, as the laws of physics imply.

Self-organized criticality explains some ubiquitous patterns existing in nature that we view as complex. Fractal structure and catastrophic events are among those regularities. Applications range from the study of pulsars and black holes to earthquakes and the evolution of life. One intriguing consequence of the theory is that catastrophes can occur for no reason whatsoever."

p. 1 "I will argue that complex behavior in nature reflects the tendency of large systems with many components to evolve into a poised, 'critical' state, way out of balance, where minor disturbances may lead to events, called avalanches, of all sizes. Most of the changes take place through catastrophic events rather than by following a smooth gradual path. The evolution to this very delicate state occurs without design from any outside agent. The state is established solely because of the dynamical interactions among individual elements of the system: the critical state is *self-organized.* Self-organized criticality is so far the only known general mechanism to generate complexity."

p. 3 "The complex phenomena observed everywhere indicate that nature operates at the self-organized critical state. The behavior of the critical sandpile mimics several phenomena observed across many sciences, which are associated with complexity."

p. 12 "…the number of earthquakes of a given magnitude follows a glaringly simple distribution function known as the Gutenberg-Richter law. It turns out that every time there are about 1,000 earthquakes of, say, magnitude 4 on the Richter scale, there are about 100 earthquakes of magnitude 5, 10 of magnitude 6, and so on. This law is illustrated in Figure 2a, …The scale is a logarithmic one, in which the number on the vertical axis are 10, 100, 1,000, instead of 1, 2, 3. The Gutenberg-Richter law manifests itself as a straight line in this plot.
The horizontal x-axis is also logarithmic …"

More examples—

(1) Mandelbrot and the price of commodities,

(2) Raup, Sepkoski and the distribution of extinction events,

(3) fractals (fjords of Norway),

(4) $1/f$ noise (water level of the Nile, light from quasars, global temperature),

(5) Zipf's law (population vs. rank of cities, frequency of use vs. rank of words).

Power Laws and Criticality

"What does it mean that something is a straight line on a double logarithmic plot? Mathematically, such straight lines are called 'power laws', since they show that some quantity N can be expressed as some power of another quantity s:

$$N(s) = s^{-\tau}.$$

Here, s could be the energy released by an earthquake and $N(s)$ could be the number of earthquakes with that energy. ... Taking the logarithm of both sides of the equation above we find

$$\log N(s) = -\tau \log s$$

This shows that $\log N(s)$ plotted versus $\log s$ is a straight line. The exponent τ is the slope of the straight line. For instance, in Zipf's law the number N of cities with more than s inhabitants was expressed as $N(s) = 1/s = s^{-1}$. That is a power law with exponent -1. Essentially all the phenomena to be discussed in this book can be expressed in terms of power laws."

"Thus, the problem of explaining the observed statistical features of complex systems can be phrased as the problem of explaining the underlying power laws, and more specifically the values of the exponents."

The Sandpile

p. 32 “The canonical example of SOC is a pile of sand.”

p. 33 “In 1987 Chao Tang, Kurt Weisenfeld, and I constructed the simple, prototypical model of self-organized criticality, the sandpile model.”

p 50 “Consider a flat table, onto which sand is added slowly, one grain at a time.... Initially, the grains of sand will stay more or less where they land. As we continue to add more sand, the pile becomes steeper, and small sand slides or avalanches occur.... ”

p 51 “Eventually the slope reaches a certain value and cannot increase any further, because the amount of sand added is balanced on average by the amount of sand leaving the pile by falling off the edges.... There will occasionally be avalanches that span the whole pile. This is the self-organized critical (SOC) state.”

"The table where sand is dropped is represented by a two-dimensional grid. At each square of the grid, with coordinates (x, y), we assign a number $Z(x, y)$, which represents the number of grains present at the square. For a table of size $L = 100$, the coordinates x and y are between 1 and 100. The total number of sites is $L \times L$. ...

The addition of a grain of sand to a square of the grid is carried out by choosing one site randomly and increasing the height Z at that site by 1:

$$Z(x, y) \to Z(x, y) + 1.$$

The process is repeated again and again. ...Whenever the height Z exceeds a critical value Z_{cr} that may arbitrarily be set, say, to 3, one grain of sand is sent to each of the four neighbors. Thus, when Z reaches 4, the height at that site decreases by 4 units,

$$Z(x, y) \to Z(x, y) - 4$$

for $Z(x, y) > Z_{cr}$, and the heights Z at the four neighbor sites go up by one unit,

$$\begin{aligned} Z(x \pm 1, y) &\to Z(x \pm 1, y) + 1, \\ Z(x, y \pm 1) &\to Z(x, y \pm 1) + 1. \end{aligned}$$

...If the unstable site happens to be at the boundary, where x or y is 1 or 100, the grains of sand simply leave the system; they fall off the edge of the table and we are not concerned with them any longer."

p. 55–6 "As the process continues, it becomes more likely that at least one of the neighbors will reach its critical height, so the first toppling event induces a second event. One toppling event leads to the next, like falling dominoes. As more sand is added, there will be bigger and bigger landslides, or avalanches, although there will still also be small ones.
Figure 12 shows a sequence of toppling events in a very small system. ...Eventually the system comes to rest. We notice that there were precisely 9 topplings, so that avalanche had size $s = 9$. ...
Eventually the sandpile enters into a stationary state where the average height of all sites does not increase further. The average height is somewhere between 2 and 3. ...
Plate 1a shows a configuration in the stationary state...".

p. 57 "The number of avalanches for a system of linear size 50 is plotted in Figure 11 on p. 47, which shows data from our very first sandpile. The straight line indicates that the avalanches follow the Gutenberg-Richter power law, just like the real earthquakes ...The power law indicates that the stationary state is critical. We conclude that the pile has self-organized into a critical state."

Power laws are signs of SOC:

p. 72 "For the long grain rice the distribution of avalanches is a power law, indicative of SOC behavior."

p. 77 "Do sand slides in nature obey the power laws indicative of SOC that were observed in the laboratory under controlled circumstances?"

p. 78 "They counted how many layers exceeded a certain thickness, and made the usual log-log histogram (Figure 19). Indeed, there is a lower law distribution of layer thickness, as the theory of SOC predicts."

p. 83 "The power law shows that the stationary state is critical."

p. 110 "The distribution was the usual power law, shown in Figure 26. The exponent, measured in the usual way as the slope of the curve, is $\tau = 1.3$. This shows that the Game of Life is critical!"

Important points:

(1) Critical states exist.

(2) Power law distributions indicate self-organized criticality.

(3) Randomness is not needed for SOC (p. 58).

(4) Systems need not be conservative to exhibit SOC (p. 93–6).

(5) Two-dimensionality is not needed for SOC (there is a linear model on p. 138).

Speculations:

p. 112 "The message is strikingly clear. The phenomena, like the formation of the 'living' structures in the Game of Life, that we intuitively identify as complex originate from a global critical dynamics. Complexity, like that of human beings, which can be observed locally in the system is the local manifestation of a globally critical process. None of the non-critical rules produce complexity. *Complexity is a consequence of criticality.*"

p. 118 "The similarity between the avalanches in the sandpile and the punctuations in evolution was astounding. Punctuations, or avalanches are the hallmark of self-organized criticality. Not long after my first visit, Stu had plotted Sepkoski's data for extinction events in the evolutionary history of life of earth the same way we had done it for the sandpiles, and found that the data were consistent with a power law, with the large extinction events occurring at the tail of the distribution (Figure 5). Could it be that biological evolution operates at the self-organized critical state?"

p. 132 “The mathematical models that Stuart Kauffman and I had studied were absurdly simplified models of evolution, and failed to capture the essential behavior . . . It turns out that the successful strategy was to make an even simpler model, rather than one that is more complicated. Insight seldom arises from complicated messy modeling, but more often from gross over-simplifications. . . . . . . it is an art to start at the complicated and messy and proceed to the simple and beautiful.”

The Bean Counter

Start with the sandpile simulation and make a few simplifications.

(1) Use only one dimension instead of two, and only one of two directions. Instead of a surface, use a line that starts from a wall.

(2) Replace sand with beans.

(3) Drop the beans only in one place. This corresponds to dribbling the sand only in the middle of the plane, rather than randomly all over the surface, as in the computer model Bak describes.

(4) Allow the system to be non-conservative. The sandpile simulation is conservative in that it retains all grains of sand (until some of them start dropping off the edge).

There is a rightmost column, with columns to the left—the exact number to the left does not matter, but say 10. Drop beans into the rightmost column until there are 10 of them. This is too many, so one of them spills into the column just to the left, called the second column. The remaining 9 beans in the first column disappear; the system is not conservative. Our story is that we just dump the beans because we are only counting the beans, not storing them. Continue to drop beans at the right end of the system, into the first column. Every tenth bean goes into the second column. When 10 beans fill the second column, one spills into the third column just to its left, and the other nine go away. Represent this model with a sequence of numbers, each telling the number of beans currently in that column. For example, if there are 7 beans in the third column, none in the second, and 8 in the third, we write 708. This situation arises exactly once, right after we have dropped 708 beans. The system records, as a positive integer in decimal notation, the number of beans dropped into the rightmost column.

Avalanches occur just as they do in the sandpile model. Using exactly the same definition as in Bak's sandpile model, we see that avalanches occur whenever the bean counter has to carry a digit to the left. For example, adding one bean to 345999 results in 34599(10), which spills over and collapses down to 3459(10)0, followed by 345(10)00, and finally 346000. This is an avalanche of magnitude 3. An avalanche of magnitude 1 occurs, for example, upon adding a bean to 89. Consider a fixed size system, with perhaps only ten columns. The final state of this system, from which we can proceed no further because there is no column to the left, is 9999999999. The life of this particular system starts with state 0000000000 and adds beans until arriving at state 9999999999. Next, count how many avalanches there are for each possible magnitude, from 1 to 9. The first avalanche of magnitude 9 occurs in the transition from 0999999999 to 1000000000, the second in the transition from 1999999999 to 2000000000, etc., with the ninth and last in the transition from 8999999999 to 9000000000. Beans added that do not cause avalanches are called "beans that cause avalanches of magnitude 0". In this way, every bean causes an avalanche of some magnitude, although that magnitude may only be zero. Here is a table showing the number of avalanches of each magnitude from 0 to 9. Note that every bean must be accounted for, so the total of the number of avalanches of each magnitude has to be the total number of beans dropped.

n is the magnitude, and $N(n)$ is the number of avalanches of magnitude n.

n	$N(n)$
9	9
8	90
7	900
6	$9,000$
5	$90,000$
4	$900,000$
3	$9,000,000$
2	$90,000,000$
1	$900,000,000$
0	$9,000,000,000$
Total	$9,999,999,999$

Just as in the sandpile model or the real world of earthquakes, magnitude is a logarithmic measure of energy. If the energy $E(n)$ in an avalanche of magnitude n is b^n, then $n = \log_b E(n)$. Plot the magnitude against the logarithm of the number of avalanches of that magnitude. The following table shows the result. $N(n)$ is the number of avalanches of magnitude n.

n	$\log N(n)$
9	$\log 9$
8	$1 + \log 9$
7	$2 + \log 9$
6	$3 + \log 9$
5	$4 + \log 9$
4	$5 + \log 9$
3	$6 + \log 9$
2	$7 + \log 9$
1	$8 + \log 9$
0	$9 + \log 9$

These data points are all *exact* solutions to the equation

$$n + \log N(n) = 9 + \log 9,$$

Since $n = \log_b E(n)$, this says

$$\log_b E(n) + \log N(n) = 9 + \log 9,$$

hence

$$N(n) = 9,000,000,000E(n))^{-\frac{1}{\log b}},$$

a power law with exponent $-\frac{1}{\log b}$. According to Bak, this proves that the bean counter is a model of self-organized criticality.

The bean counter is a familiar object to those who drive cars and trucks. It's called an

ODOMETER

Odometers are models of self-organized criticality.

A speculation from Bak, p. 118, edited:

The similarity between the avalanches in the sandpile and carrying in addition was astounding. Carrying digits, or avalanches are the hallmark of self-organized criticality. After reading my book, Rog has plotted data for carrying events in the additive history of numbers of beans the same way we had done it for the sandpiles, and found that the data were consistent with a power law, with the large carrying events occurring at the tail of the distribution. Could it be that odometers operate at the self-organized critical state?

it was not natural to have come from there
yes
write about if I like or anything if I like
but not there,
there is no there there

Everybody's Autobiography
Gertrude Stein (1874-1946)

What would Mr. Natural say?