RELEVANCE LOGIC AND THE CALCULUS OF RELATIONS

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Abstract. Sound and complete semantics for classical propositional logic can be obtained by interpreting sentences as sets. Replacing sets with commuting dense binary relations produces an interpretation that turns out to be sound but not complete for **R**. Adding transitivity yields sound and complete semantics for **RM**, because all normal Sugihara matrices are representable as algebras of binary relations.

1. Introduction

One way to get sound and complete semantics for classical propositional logic is to evaluate each variable as one of two truth values, and extend this valuation to more complicated sentences by the classical truth tables. Another way to get sound and complete semantics for classical propositional logic is to evaluate each variable as a subset of a fixed universe of discourse. For complex sentences, interpret conjunction as intersection, disjunction as union, negation as complementation, and so on. These two methods are essentially the same, but the second one provides an obvious generalization: replace "set" with "binary relation". This approach was taken by Tarski, who produced an undecidable fragment of classical propositional logic by early 1942; see (Tarski and Givant 1987, §5.4, §5.5, fn. 3*). Tarski's operations include Boolean intersection $\cap$, union $\cup$, and complementation $\bar{\ }$, relative (or Peircean) multiplication $|$ and addition $\dagger$, conversion $^{-1}$, and an identity relation.

Relevance logic arose in the 1950s and 1960s from attempts to axiomatize the notion that an implication $A \to B$ should be regarded as true only if the hypothesis A is "relevant" to the conclusion B. The earliest systems were proposed by Orlov in 1928 (Došen 1992), and by Moh (1950), Church (1951), and Ackermann (1956) in the 1950s. Semantics were introduced and developed only much later, in the 1970s; see (Routley and Routley 1972), (Routley and Meyer 1972a), (Routley and Meyer 1972b), (Urquhart 1972), (Routley and Meyer 1973), (Fine 1974), (Meyer and Routley 1973), (Anderson and Belnap 1975), (Routley, Plumwood, Meyer, and Brady 1982), (Anderson, Belnap, and Dunn 1992), and (Brady 2003).

The calculus of relations was created by De Morgan (1856), (1864a), (1864b), (1966) and Peirce (1870), (1880), (1883), (1885), (1897), (1960), (1984), and was extensively developed by Schröder (1966). Relation algebras arose from Tarski's axiomatization of the calculus of relations; see (Tarski 1941), (Tarski and Givant 1987) and (Maddux 1991). Tarski's undecidable propositional calculus is equivalent to the equational theory of relation algebras.

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The Routley-Meyer semantics for relevance logic and the theory of relation algebras have a significant class of structures in common. A structure is in this class if it is simultaneously the atom structure of a relation algebra and a normal relevant model structure. Prominent examples of these are the ones constructed by Lyndon (1961) from projective planes. This connection is the key to deep undecidability results in both subjects; see (Andréka, Givant, and Németi 1997) and (Urquhart 1984).

This confluence makes it possible to think of propositional variables, sentences, and worlds in a relevant model structure as binary relations. The connectives of relevance logic are then certain operations on binary relations determined by the Routley-Meyer semantics. For example, negation $\sim$ turns out to be converse-complementation while fusion $\circ$ is simply composition. The constants of relevance logic will not be considered here because they are the source of some difficulties; see (Routley, Plumwood, Meyer, and Brady 1982, p. 348), (Bimbo, Dunn, and Maddux 2008).

In Section 2 we present axioms and rules of deduction for relevance logic, and focus attention on two prominent systems, **R** and **RM**. Sections 3 and 4 introduce relational relevance algebras and give two examples, due to Belnap and Meyer. Soundness for the interpretation of sentences as binary relations is shown in Section 5. In Section 6 we prove that **RM** is a complete axiomatization of the logic of transitive commutative dense relational relevance algebras, while in Sections 7 and 8 we show that **R** is an incomplete axiomatization of the logic of commutative dense relational relevance algebras. Some closing remarks are made in Section 9.

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2. Systems of relevance logic

Let Pv be a countable set whose elements are called **propositional variables**. There are five **connectives**, $\vee$, $\wedge$, $\circ$, $\rightarrow$, and $\sim$. For any $C \subseteq \{\vee, \wedge, \circ, \rightarrow, \sim\}$, the set Sent_C of C-**sentences** is the closure of the variables under application of the connectives in C. Let $\mathsf{Sent} := \mathsf{Sent}_{\{\vee,\wedge,\circ,\rightarrow,\sim\}}$. The connectives are operations on Sent which act in the way required of a language, that is, $\langle \mathsf{Sent}, \vee, \wedge, \circ, \rightarrow, \sim\rangle$ is an algebra of type $\langle 2, 2, 2, 2, 1\rangle$ (four binary operations and one unary operation) which is **absolutely freely generated** by Pv. This means that $\langle \mathsf{Sent}, \vee, \wedge, \circ, \rightarrow, \sim\rangle$ is generated by Pv and any function from Pv to an algebra $\mathfrak{R}$ of type $\langle 2, 2, 2, 2, 1\rangle$ has a unique extension to a homomorphism from $\langle \mathsf{Sent}, \vee, \wedge, \circ, \rightarrow, \sim\rangle$ into $\mathfrak{R}$.

A sentence $S \in \mathsf{Sent}$ is an **axiom of R** if there are sentences $A, B, C \in \mathsf{Sent}$ such that S is one of the sentences (A1)–(A31) listed below, and S an **axiom of RM** if S is one of (A1)–(A33). The numbering of (Routley and Meyer 1973, pp. 204, 224) is on the right.

(A1)	$A \rightarrow A$	A1
(A2)	$A \wedge B \rightarrow A$	A5
(A3)	$A \wedge B \rightarrow B$	A6
(A4)	$((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$	A7
(A5)	$A \rightarrow A \vee B$	A8

(A6)	$B \to A \vee B$	A9
(A7)	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	A10
(A8)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	A11
(A9)	$\sim\sim A \to A$	A13
(A10)	$\sim(A \vee B) \to (\sim A \wedge \sim B)$	
(A11)	$(\sim A \wedge \sim B) \to \sim(A \vee B)$	
(A12)	$(A \to B) \to ((C \to A) \to (C \to B))$	
(A13)	$((A \to A) \to B) \to B$	
(A14)	$A \to ((\sim B \to \sim A) \to B)$	
(A15)	$A \to (\sim B \to \sim(A \to B))$	
(A16)	$(A \to (B \to C)) \to (\sim(A \to \sim B) \to C)$	
(A17)	$A \circ B \to \sim(A \to \sim B)$	
(A18)	$\sim(A \to \sim B) \to A \circ B$	
(A19)	$A \to (B \to (A \circ B))$	A14
(A20)	$(A \to (B \to C)) \to ((A \circ B) \to C)$	A15
(A21)	$(A \to (B \to C)) \to (B \to (A \to C))$	
(A22)	$(A \to \sim B) \to (B \to \sim A)$	A12
(A23)	$A \to ((A \to B) \to B)$	A2
(A24)	$(A \to B) \to ((B \to C) \to (A \to C))$	A3
(A25)	$(A \to (A \to B)) \to (A \to B)$	A4
(A26)	$(A \to \sim A) \to \sim A$	
(A27)	$(A \to (B \to C)) \to ((A \wedge B) \to C)$	
(A28)	$(A \to B) \to (\sim A \vee B)$	
(A29)	$(A \wedge (A \to B)) \to B$	
(A30)	$((A \to B) \wedge (B \to C)) \to (A \to C)$	
(A31)	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	
(A32)	$A \to (A \to A)$	
(A33)	$(A \to B) \to (A \to (A \to B))$	

Among the following rules of deduction, only *modus ponens* and Adjunction are used in **R** and **RM**. The rules used in the Basic Logic of (Routley, Plumwood, Meyer, and Brady 1982, p. 287) are *modus ponens*, Adjunction, Suffixing, Prefixing, and Contraposition.

$A,\ A \to B \vdash B$	***modus ponens***
$A,\ B \vdash A \wedge B$	**Adjunction**
$A \to \sim B \vdash B \to \sim A$	**Contraposition**
$\sim A,\ A \vee B \vdash B$	**Disjunctive Syllogism**
$A \wedge B \to C,\ B \to C \vee A \vdash B \to C$	**Cut**
$A \to B \vdash (C \to A) \to (C \to B)$	**Prefixing**

$A \to B \vdash (B \to C) \to (A \to C)$ **Suffixing**

$A \to (B \to C) \vdash B \to (\sim C \to \sim A)$ **Cycling**

$A \vdash (A \to B) \to B$ **E-rule** (Brady 2003, p. 8)

For any $A \in \mathsf{Sent}$, we write $\vdash_{\mathbf{R}} A$ (or $\vdash_{\mathbf{RM}} A$) if A belongs to every subset of Sent that contains the axioms of **R** (or **RM**) and is closed under *modus ponens* and Adjunction. This axiomatization of **R** is highly redundant but provides more input for semantic analysis in Theorem 4. Routley and Meyer (1973) use only A1–A15. Furthermore, **R** is well-axiomatized in the following sense.

Theorem 1 (Routley and Meyer 1973, Th. 7)**.** *Let C be one of the following sets of connectives:*

$$\{\to\}, \{\to,\sim\}, \{\to,\circ\}, \{\to,\sim,\circ\}, \{\to,\wedge\}, \{\to,\vee,\wedge\},$$
$$\{\to,\circ,\wedge\}, \{\to,\circ,\wedge,\vee\}, \{\to,\sim,\wedge,\vee\}, \{\to,\sim,\circ,\wedge,\vee\}.$$

If $A \in \mathsf{Sent}_C$, then $\vdash_{\mathbf{R}} A$ iff A is derivable, using only modus ponens and Adjunction, from those axioms among A1–A15 *that explicitly contain connectives in C.*

3. RELATIONAL RELEVANCE ALGEBRAS

Binary relations are, by definition, sets of ordered pairs. For arbitrary binary relations A and B, their union, intersection, difference, converse, and relative product are defined as follows.

(1) $$A \cup B := \{\langle x,y\rangle : \langle x,y\rangle \in A \text{ or } \langle x,y\rangle \in B\}$$

(2) $$A \cap B := \{\langle x,y\rangle : \langle x,y\rangle \in A \text{ and } \langle x,y\rangle \in B\}$$

(3) $$A - B := \{\langle x,y\rangle : \langle x,y\rangle \in A \text{ and } \langle x,y\rangle \notin B\}$$

(4) $$A^{-1} := \{\langle x,y\rangle : \langle y,x\rangle \in A\}$$

(5) $$A|B := \{\langle x,y\rangle : \exists_z(\langle x,z\rangle \in A \text{ and } \langle z,y\rangle \in B)\}$$

Let U be a non-empty set. $U^2 = \{\langle x,y\rangle : x,y \in U\}$ is the set of ordered pairs of elements of U. $Sb\left(U^2\right)$ is the set of subsets of U^2, and is called the **set of binary relations on** U. The identity and diversity relations on U are

(6) $$\mathsf{Id} := \{\langle x,x\rangle : x \in U\}$$

(7) $$\mathsf{Di} := \{\langle x,y\rangle : x,y \in U \text{ and } x \neq y\}$$

For any binary relations $A, B \subseteq U^2$, set

(8) $\overline{A} := U^2 - A$ (Boolean complement)

(9) $\sim A := U^2 - A^{-1}$ (De Morgan complement)

(10) $A \circ B := B|A$ (composition)

(11) $A \dagger B := \sim(\sim A \circ \sim B)$ (relative sum)

(12) $A \to B := \sim(\sim A \circ B)$ (residual)

Alternate characterizations, obtained by unwinding definitions, are

$$\overline{A} = \{\langle x,y\rangle : x,y \in U \text{ and } \langle x,y\rangle \notin A\}$$
$$\sim A = \{\langle x,y\rangle : x,y \in U \text{ and } \langle y,x\rangle \notin A\}$$
$$A \dagger B = \{\langle x,y\rangle : x,y \in U \text{ and } \forall_{z\in U}(\langle x,z\rangle \in A \text{ or } \langle z,y\rangle \in B)\}$$

$$A \to B = \{\langle x, y\rangle : x, y \in U \text{ and } \forall_{z\in U}(\text{ if } \langle z, x\rangle \in A \text{ then } \langle z, y\rangle \in B)\}$$

A **relational relevance algebra on** a non-empty set U is an algebra

$$\mathfrak{R} = \langle R, \cup, \cap, \circ, \to, \sim\rangle \tag{13}$$

of type $\langle 2, 2, 2, 2, 1\rangle$ such that R is a non-empty set of relations on U, and R is closed under the operations $\cup$, $\cap$, $\circ$, $\to$, and $\sim$. For example, choosing $R = Sb\left(U^2\right)$ produces the relational relevance algebra of all binary relations on the set U,

$$\mathfrak{Rel}(U) := \left\langle Sb\left(U^2\right), \cup, \cap, \circ, \to, \sim\right\rangle.$$

Relational relevance algebras lack the constants of relevant algebras (Urquhart 1996) or De Morgan monoids (Anderson and Belnap 1975), but they do satisfy many equations not involving constants that have been used in the definitions of these and other algebras designed for relevance logic. For example, if $\mathfrak{R} = \langle R, \cup, \cap, \circ, \to, \sim\rangle$ is a relational relevance algebra, then $\langle R, \cup, \cap\rangle$ is a distributive lattice, $\langle R, \circ\rangle$ is a semigroup, and many other equations and inclusions hold for all $A, B, C \in R$, such as

$$\begin{aligned}
A \circ (B \cup C) &= (A \circ B) \cup (A \circ C)\\
(B \cup C) \circ A &= (B \circ A) \cup (C \circ A)\\
(A \cup B) \to C &= (A \to B) \cap (A \to C)\\
A \to (B \cap C) &= (A \to B) \cap (A \to C)\\
(A \cup B) \to C &= (A \to C) \cap (B \to C)\\
\sim(\sim A) &= A\\
\sim(A \cup B) &= \sim A \cap \sim B\\
\sim(A \cap B) &= \sim A \cup \sim B\\
A \circ B &= \sim(A \to \sim B)\\
A \to B &= \sim(A \circ \sim B)\\
A \to (B \to C) &= A \circ B \to C\\
(A \to B) \circ A &\subseteq B\\
A \to B &\subseteq (C \to A) \to (C \to B)
\end{aligned}$$

If R is a set of relations closed under composition, we say that R is **commutative** if $A \circ B = B \circ A$ for every $A, B \in R$. A relational relevance algebra $\mathfrak{R}$ is **commutative** if its universe R is commutative. For example, $\mathfrak{Rel}(U)$ is commutative iff $|U| = 1$. In a commutative relational relevance algebra, $A \to \sim B = B \to \sim A$ and $A \to B \subseteq (B \to C) \to (A \to C)$.

A binary relation A is **dense** if $A \subseteq A \circ A$, **transitive** if $A \circ A \subseteq A$, and **symmetric** if $A = A^{-1}$. We say that a relational relevance algebra $\mathfrak{R}$ is **dense**, **transitive**, or **symmetric** if every relation in $\mathfrak{R}$ is dense, transitive, or symmetric, respectively. Let R, R^{cd}, and $\mathsf{R}^{\mathsf{cdt}}$ be the classes of relational relevance algebras, commutative dense relational relevance algebras, and commutative dense transitive relational relevance algebras, respectively. Define R^{c}, R^{d}, R^{dt}, R^{ct}, and R^{t} similarly. For any class S of algebras let $\underline{\mathbf{I}}S$ be the class of algebras isomorphic to algebras in S.

Symmetry is preserved by $\sim$, for if A is a symmetric relation then $\sim A$ is also symmetric. However, transitivity is not preserved by $\sim$ because, for any non-empty

U, Id is transitive but $\sim\mathsf{Id}$ is not transitive. It follows that no transitive relational relevance algebra on a non-empty U contains the identity relation on U. The identity relation Id is always dense, but $\sim\mathsf{Id}$ is not dense whenever $|U| = 2$. This is one of the reasons for not requiring Id to belong to a relational relevance algebra.

No relational relevance algebra has any $\sim$-fixed points, for if $A \subseteq U^2$ and $A = \sim A$, then $\langle x, y\rangle \in A$ iff $\langle y, x\rangle \notin A$ for all $x, y \in U$, hence $\langle x, x\rangle \in A$ iff $\langle x, x\rangle \notin A$, a contradiction.

A relational relevance algebra generated by a commutative set of relations may not be commutative. For example, if $U = \{0, 1\}$, $B = \emptyset$, and $A = \{\langle 0, 1\rangle\}$, then $\sim B = U^2$ and $A \circ B = B \circ A = \emptyset$, so $\{A, B\}$ is commutative, but

$$\{0\} \times U = U^2 \circ A = \sim B \circ A \neq A \circ \sim B = A \circ U^2 = U \times \{1\}$$

so $\{A, \sim B\}$ is not commutative.

Suppose $\mathfrak{R}$ is a relational relevance algebra on U and Id is the identity relation on U. We say that a sentence $A \in \mathsf{Sent}$ is **valid in** $\mathfrak{R}$, and write $\mathfrak{R} \models A$, if $\mathsf{Id} \subseteq H(A)$ for every homomorphism H from $\langle \mathsf{Sent}, \vee, \wedge, \circ, \to, \sim\rangle$ into $\mathfrak{R}$. For any class $S \subseteq \mathsf{R}$ of relational relevance algebras, A is **valid in** S if A is valid in every algebra in S, and S**-logic** is the set of sentences valid in S. The notion of validity applies to isomorphic copies of relational relevance algebras in the obvious way, so S-logic is the same as $\underline{\mathbf{I}}S$-logic.

4. Relational relevance algebras of Belnap and Meyer

In this section we give two useful examples of relational relevance algebras, one on an infinite set, and one on a finite set. For these examples we first define two closely related finite algebras, Belnap's $\mathbf{M}_0$ and Meyer's $\mathbf{RM}84$. They can be defined together as follows.

(i) Both $\mathbf{M}_0$ and $\mathbf{RM}84$ are algebras of the form $\langle S_3, \vee, \wedge, \circ, \to, \sim\rangle$ where

$$S_3 := \{-3, -2, -1, -0, +0, +1, +2, +3\},$$

the set of designated values is $\{+0, +1, +2, +3\}$, and a sentence A is **valid in** the algebra if every homomorphism from the algebra of sentences carries A to a designated value.

(ii) For both algebras the reduct $\langle S_3, \vee, \wedge\rangle$ is the lattice of a Boolean algebra whose atoms are -1, $+0$, and -2, whose top element is $+3$ and bottom element is -3, satisfying these equations: $-1 \vee +0 = +1$, $-1 \vee -2 = -0$, and $+0 \vee -2 = +2$. For tables see (Belnap 1960, p. 145), (Anderson and Belnap 1975, p. 252), (Routley, Plumwood, Meyer, and Brady 1982, pp. 178, 253), and (Brady 2003, p. 101); for Hasse diagrams see (Anderson and Belnap 1975, pp. 198, 252), (Routley, Plumwood, Meyer, and Brady 1982, p. 178), and (Brady 2003, p. 102).

(iii) In both algebras the operation $\sim$ takes $-i$ to $+i$ and $+i$ to $-i$ for every $i \in \{0, 1, 2, 3\}$.

(iv) The operation $\to$ in Belnap's $\mathbf{M}_0$ is defined in Table 1, (Belnap 1960, p. 145), (Anderson and Belnap 1975, p. 253), and (Brady 2003, p. 101).

(v) The operation $\to$ in Meyer's $\mathbf{RM}84$ is defined in Table 2, (Anderson and Belnap 1975, p. 334), and (Routley, Plumwood, Meyer, and Brady 1982, p. 253).

(vi) In both algebras the operation $\circ$ is defined by $x \circ y = \sim(x \to \sim y)$; see Tables 3 and 4.

$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+3$
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+3$
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+3$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+3$
$+3$	-3	-3	-3	-3	-3	-3	-3	$+3$

TABLE 1. Table for the operation $\to$ in $\mathbf{M}_0$

$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
-2	-3	$+0$	-3	$+2$	-3	-3	$+0$	$+3$
-1	-3	-3	$+0$	$+1$	-3	$+0$	-3	$+3$
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
$+1$	-3	-3	-3	-1	-3	$+0$	-3	$+3$
$+2$	-3	-3	-3	-2	-3	-3	$+0$	$+3$
$+3$	-3	-3	-3	-3	-3	-3	-3	$+3$

TABLE 2. Table for the operation $\to$ in $\mathbf{RM}84$

$\circ$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
-3	-3	-3	-3	-3	-3	-3	-3	-3
-2	-3	-2	$+3$	$+3$	-2	$+3$	-2	$+3$
-1	-3	$+3$	-1	$+3$	-1	-1	$+3$	$+3$
-0	-3	$+3$	$+3$	$+3$	-0	$+3$	$+3$	$+3$
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
$+1$	-3	$+3$	-1	$+3$	$+1$	$+1$	$+3$	$+3$
$+2$	-3	-2	$+3$	$+3$	$+2$	$+3$	$+2$	$+3$
$+3$	-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$

TABLE 3. Table for the operation $\circ$ in $\mathbf{M}_0$

In the next two theorems we show that $\mathbf{M}_0$ and $\mathbf{RM}84$ are isomorphic to algebras in R^{cd}, and hence belong to $\underline{\mathbf{I}}\mathsf{R}^{\mathsf{cd}}$. Theorem 2 was announced in the abstract (Maddux 2007) and noted again in (Bimbo, Dunn, and Maddux 2008), while Theorem 3 is new.

Theorem 2. *Belnap's $\mathbf{M}_0$ is isomorphic to a commutative dense relational relevance algebra on a countable set, so*

$$\mathbf{M}_0 \in \underline{\mathbf{I}}\mathsf{R}^{\mathsf{cd}}. \tag{14}$$

$\circ$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
-3	-3	-3	-3	-3	-3	-3	-3	-3
-2	-3	-0	$+3$	$+3$	-2	$+3$	-0	$+3$
-1	-3	$+3$	-0	$+3$	-1	-0	$+3$	$+3$
-0	-3	$+3$	$+3$	$+3$	-0	$+3$	$+3$	$+3$
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
$+1$	-3	$+3$	-0	$+3$	$+1$	$+3$	$+3$	$+3$
$+2$	-3	-0	$+3$	$+3$	$+2$	$+3$	$+3$	$+3$
$+3$	-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$

TABLE 4. Table for the operation $\circ$ in **RM**84

Proof. Let $\mathbb{Q}$ be the set of rational numbers. Define a map ρ from the universe S_3 of $\mathbf{M}_0$ into the set of binary relations on $\mathbb{Q}$, as follows.

$$\begin{aligned}
\rho(-3) &:= \emptyset,\\
\rho(-2) &:= \{\langle x,y\rangle : x\in\mathbb{Q}, x>y\in\mathbb{Q}\},\\
\rho(-1) &:= \{\langle x,y\rangle : x\in\mathbb{Q}, x<y\in\mathbb{Q}\},\\
\rho(-0) &:= \rho(-1)\cup\rho(-2),\\
\rho(+0) &:= \mathsf{Id} := \{\langle x,x\rangle : x\in\mathbb{Q}\},\\
\rho(+1) &:= \rho(-1)\cup\rho(+0),\\
\rho(+2) &:= \rho(-2)\cup\rho(+0),\\
\rho(+3) &:= \mathbb{Q}^2.
\end{aligned}$$

Then $\rho(S_3)$ is closed under $\cup$, $\cap$, $\circ$, $\to$, and $\sim$, $\langle\rho(S_3),\cup,\cap,\circ,\to,\sim\rangle$ is a commutative dense relational relevance algebra, ρ is an isomorphism, and

$$\mathbf{M}_0 \cong \langle\rho(S_3),\cup,\cap,\circ,\to,\sim\rangle \in \mathsf{R}^{\mathsf{cd}}.$$

□

For every $i\in S_3$, i is a designated value iff $\mathsf{Id}\subseteq\rho(i)$. Therefore a sentence A is valid in $\mathbf{M}_0$ according to its definition as an algebra with designated values iff A is valid in $\mathbf{M}_0$ as a relational relevance algebra. The same is true for **RM**84.

Theorem 3. *Meyer's* **RM**84 *is isomorphic to a commutative dense relational relevance algebra on a 7-element set, so*

$$\mathbf{RM}84 \in \underline{\mathsf{I}}\mathsf{R}^{\mathsf{cd}}. \tag{15}$$

Proof. Define a map ρ from the universe S_3 of **RM**84 to the set of binary relations on $U := \{0,1,2,3,4,5,6\}$, where "$+_7$" denotes addition *modulo* 7:

$$\begin{aligned}
\rho(-3) &:= \emptyset,\\
\rho(-2) &:= \{\langle x, x+_7 y\rangle : x\in U, y\in\{3,5,6\}\},\\
\rho(-1) &:= \{\langle x, x+_7 y\rangle : x\in U, y\in\{1,2,4\}\},\\
\rho(-0) &:= \rho(-1)\cup\rho(-2),\\
\rho(+0) &:= \{\langle x,x\rangle : x\in\{0,\ldots,6\}\},\\
\rho(+1) &:= \rho(-1)\cup\rho(+0),
\end{aligned}$$

$$\rho(+2) := \rho(-2) \cup \rho(+0),$$
$$\rho(+3) := U^2.$$

Then $\rho(S_3)$ is closed under $\cup$, $\cap$, $\circ$, $\to$, and $\sim$, $\langle \rho(S_3), \cup, \cap, \circ, \to, \sim \rangle$ is a commutative dense relational relevance algebra, ρ is an isomorphism, and

$$\mathbf{RM}84 \cong \langle \rho(S_3), \cup, \cap, \circ, \to, \sim \rangle \in \mathsf{R}^{\mathsf{cd}}.$$

□

The logic called **BM** is defined by Brady (2003, p. 128) as an extension of **R**. Brady (2003, p. 138) proves that Belnap's $\mathbf{M}_0$ is characteristic for the logic **BM**, so **BM** is $\{\mathbf{M}_0\}$-logic. By Theorem 2 we have $\{\mathbf{M}_0\} \subseteq \underline{\mathbf{I}}\mathsf{R}^{\mathsf{cd}}$, hence **BM** is a complete decidable extension of R^{cd}-logic. By Theorem 3, $\{\mathbf{RM}84\}$-logic is a complete decidable extension of R^{cd}-logic.

$\mathbf{M}_0$ and $\mathbf{RM}84$ may be replaced, for all algebraic purposes, with their relational descriptions. Instead of eight elements with operations defined on them by tables, we have eight relations with set-theoretically defined operations: intersection, composition, etc. For example, if we let A be the relation $<$ on the rationals $\mathbb{Q}$ and let B be the relation $>$ on $\mathbb{Q}$, then a simple calculation shows $\emptyset = A \to (B \to A) = (A \cap \sim A) \to B$, so the sentences $A \to (B \to A)$ and $(A \wedge \sim A) \to B$ are not provable in **R**. For another such proof, of a more general result, first note that $\{<, \leq\}$ and $\{>, \geq\}$ are closed under $\cup, \cap, \circ, \to, \sim$, and $A \to B = \emptyset$ whenever $A \in \{<, \leq\}$ and $B \in \{>, \geq\}$. Suppose the sentences A and B share no variable. By evaluating the variables of A as $<$ and the variables of B as $>$, we get $A \in \{<, \leq\}$ and $B \in \{>, \geq\}$, hence $A \to B = \emptyset$. It follows that $A \to B$ is not a theorem of **R**.

5. Soundness

The Peirce-Schröder calculus of relations may be defined as Boolean combinations of equations between terms denoting relations. The terms are built up from variables using complementation $\overline{}$, intersection $\cap$, union $\cup$, relative multiplication $|$, composition $\circ$, relative addition $\dagger$, conversion $^{-1}$, and the identity relation Id. Relevance logic is the fragment of the calculus of relations in which the terms are built up using only intersection $\cap$, union $\cup$, residuation $\to$, and converse-complementation $\sim$. Relative multiplication and composition are definable in this fragment since $A|B = \sim(B \to \sim A)$ and $A \circ B = \sim(A \to \sim B)$, so one may understand relevance logic as the restriction of the calculus of relations to the operations $\cup$, $\cap$, $|$, $\circ$,$\to$, and $\sim$.

Schröder (1966, §11, pp. 153ff) showed that if a term is understood as the assertion that the relation it denotes contains the universal relation, then every Boolean combination of equations between terms denoting relations is equivalent to a single term. This convention allows the formulation of the calculus of relations as a sentential calculus; for details see (Tarski and Givant 1987, Ch. 5). The corresponding convention for relevance logic is that an individual term asserts that the relation it denotes contains the identity relation.

In the next theorem, parts (16)–(22) are handy computational rules, parts (23)–(31) show that validity is preserved in all relational relevance algebras by the rules of deduction, parts (32)–(47) show several sentences are valid in R, and the remaining parts give sentences valid in R^{c}, R^{d}, R^{cd}, and R^{t}.

Theorem 4. *Suppose U is a set and $A, B, C, D, E, F, G \subseteq U^2$. Then*

(16) $\mathsf{Id} \to A = A$,

(17) $A \subseteq B$ *iff* $\mathsf{Id} \subseteq A \to B$,

(18) $A \to (B \to C) = B|A \to C = A \circ B \to C$,

(19) $A|(A \to B) \subseteq B, \quad (A \to B) \circ A \subseteq B$,

(20) $(A \to B)|{\sim}B \subseteq {\sim}A, \quad {\sim}B \circ (A \to B) \subseteq {\sim}A$,

(21) $A \subseteq B$ *implies* $B \to C \subseteq A \to C$,

(22) $A \subseteq B$ *implies* $C \to A \subseteq C \to B$.

The rules of deduction preserve validity in R *because*

(23) *if* $\mathsf{Id} \subseteq A$ *and* $\mathsf{Id} \subseteq A \to B$ *then* $\mathsf{Id} \subseteq B$,

(24) *if* $\mathsf{Id} \subseteq A$ *and* $\mathsf{Id} \subseteq B$ *then* $\mathsf{Id} \subseteq A \cap B$,

(25) *if* $\mathsf{Id} \subseteq A \to {\sim}B$ *then* $\mathsf{Id} \subseteq B \to {\sim}A$,

(26) *if* $\mathsf{Id} \subseteq {\sim}A$ *and* $\mathsf{Id} \subseteq A \cup B$ *then* $\mathsf{Id} \subseteq B$,

(27) *if* $\mathsf{Id} \subseteq A \cap B \to C$ *and* $\mathsf{Id} \subseteq B \to C \cup A$ *then* $\mathsf{Id} \subseteq B \to C$,

(28) *if* $\mathsf{Id} \subseteq A \to B$ *then* $\mathsf{Id} \subseteq (C \to A) \to (C \to B)$,

(29) *if* $\mathsf{Id} \subseteq A \to B$ *then* $\mathsf{Id} \subseteq (B \to C) \to (A \to C)$,

(30) *if* $\mathsf{Id} \subseteq A \to (B \to C)$ *then* $\mathsf{Id} \subseteq B \to ({\sim}C \to {\sim}A)$,

(31) *if* $\mathsf{Id} \subseteq A$ *then* $\mathsf{Id} \subseteq (A \to B) \to B$,

(A1)–(A20) *are valid in* R *because*

(32) $\mathsf{Id} \subseteq A \to A$,

(33) $\mathsf{Id} \subseteq A \cap B \to A$,

(34) $\mathsf{Id} \subseteq A \cap B \to B$,

(35) $\mathsf{Id} \subseteq ((A \to B) \cap (A \to C)) \to (A \to (B \cap C))$,

(36) $\mathsf{Id} \subseteq A \to A \cup B$,

(37) $\mathsf{Id} \subseteq B \to A \cup B$,

(38) $\mathsf{Id} \subseteq ((A \to C) \cap (B \to C)) \to ((A \cup B) \to C)$,

(39) $\mathsf{Id} \subseteq A \cap (B \cup C) \to (A \cap B) \cup (A \cap C)$,

(40) $\mathsf{Id} \subseteq {\sim}{\sim}A \to A$,

(41) $\mathsf{Id} \subseteq {\sim}(A \cup B) \to ({\sim}A \cap {\sim}B)$,

(42) $\mathsf{Id} \subseteq ({\sim}A \cap {\sim}B) \to {\sim}(A \cup B)$,

(43) $\mathsf{Id} \subseteq (A \to B) \to ((C \to A) \to (C \to B))$,

(44) $\mathsf{Id} \subseteq ((A \to A) \to B) \to B$,

(45) $\mathsf{Id} \subseteq A \to (({\sim}B \to {\sim}A) \to B)$,

(46) $\mathsf{Id} \subseteq A \to ({\sim}B \to {\sim}(A \to B))$,

(47) $\mathsf{Id} \subseteq (A \to (B \to C)) \to ({\sim}(A \to {\sim}B) \to C)$,

(48) $\mathsf{Id} \subseteq A \circ B \to {\sim}(A \to {\sim}B)$,

(49) $\mathsf{Id} \subseteq \sim(A \to \sim B) \to A \circ B$,

(50) $\mathsf{Id} \subseteq A \to (B \to (A \circ B))$,

(51) $\mathsf{Id} \subseteq (A \to (B \to C)) \to ((A \circ B) \to C)$,

(A21)–(A24) *are valid in* R^{c} *because*

(52) *if* $\{A, B\}$ *is commutative then* $\mathsf{Id} \subseteq (A \to (B \to C)) \to (B \to (A \to C))$,

(53) *if* $\{A, B\}$ *is commutative then* $\mathsf{Id} \subseteq (A \to \sim B) \to (B \to \sim A)$,

(54) *if* $\{A, A \to B\}$ *is commutative then* $\mathsf{Id} \subseteq A \to ((A \to B) \to B)$,

(55) *if* $\{B \to C, A \to B\}$ *is commutative, then*
$\mathsf{Id} \subseteq (A \to B) \to ((B \to C) \to (A \to C))$,

(A25)–(A30) *are valid in* R^{d} *because*

(56) *if* A *is dense then* $\mathsf{Id} \subseteq (A \to (A \to B)) \to (A \to B)$,

(57) *if* A *is dense then* $\mathsf{Id} \subseteq (A \to \sim A) \to \sim A$,

(58) *if* $A \cap B$ *is dense then* $\mathsf{Id} \subseteq (A \to (B \to C)) \to ((A \cap B) \to C)$

(59) *if* $A \cap \sim B$ *is dense then* $\mathsf{Id} \subseteq (A \to B) \to (\sim A \cup B)$

(60) *if* $A \cap (A \to B)$ *is dense then* $\mathsf{Id} \subseteq (A \cap (A \to B)) \to B$

(61) *if* $(A \to B) \cap (B \to C)$ *is dense then*
$\mathsf{Id} \subseteq ((A \to B) \cap (B \to C)) \to (A \to C)$

(A31) *is valid in* R^{cd} *because*

(62) *if* $\{A, A \to B\}$ *is commutative and* A *is dense then*
$\mathsf{Id} \subseteq (A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(A32) *and* (A33) *are valid in* R^{t} *because*

(63) *if* A *is transitive then* $\mathsf{Id} \subseteq A \to (A \to A)$,

(64) *if* A *is transitive then* $\mathsf{Id} \subseteq (A \to B) \to (A \to (A \to B))$.

Prefixing is valid in R both as a rule and as axiom (A12).

Suffixing is valid in R as a rule but not as axiom (A24), because there are non-commutative relational relevance algebras. On the other hand, (A24) holds in a relational relevance algebra whenever $A \to B$ and $B \to C$ commute, so (A24) is valid in R^{c}. This does not seem to exclude the possibility of a non-commutative relational relevance algebra in which (A24) is valid.

Contraposition is valid as a rule in R but not as axiom (A22), again because non-commutative relational relevance algebras exist. (A21) and (A22) are valid in R^{c}; in fact, they are valid in a relational relevance algebra $\mathfrak{R}$ iff $\mathfrak{R}$ is commutative.

We can use relations to give independence proofs. For example, the existence of non-commutative relational relevance algebras shows that (A21)–(A24) cannot be proved from axioms (A1)–(A20) using all nine rules.

Corollary 5. (Soundness Theorem) *For every* $A \in \mathsf{Sent}$,

(i) *if* $\vdash_{\mathbf{R}} A$ *then* A *is valid in* R^{cd},

(ii) *if* $\vdash_{\mathbf{RM}} A$ *then* A *is valid in* $\mathsf{R}^{\mathsf{cdt}}$.

Two questions were asked in (Maddux 2007):

(Q1) if A is not a theorem of **R**, is there some $\mathfrak{R} \in \mathsf{R}^{\mathsf{cd}}$ in which A is not valid?
(Q2) if A is not a theorem of **RM**, is there some $\mathfrak{R} \in \mathsf{R}^{\mathsf{cdt}}$ in which A is not valid?

The (expected) answer to (Q1) is "no". This was first established by Mikulás (2008), who proved that there is no finite axiomatization of R^{cd}-logic. In Section 8 we present two examples of sentences in R^{cd}-logic that are not theorems of **R**. The (unexpected) answer to (Q2) is "yes", for reasons given in the next section.

6. Completeness of **RM** for $\mathsf{R}^{\mathsf{cdt}}$

Sugihara matrices were introduced by Sugihara (1955) and simplified by Anderson and Belnap (1975, §26.9). R. K. Meyer used them to prove completeness results for **RM**; see (Anderson and Belnap 1975, §29.3).

We define only the finite Sugihara matrices $\mathbf{S}_n$, with $2 \leq n < \omega$. If $n = 2k$ for some $k > 0$ then

$$S_n := \{-k, \ldots, -1, 1, \ldots, k\}$$

with designated values $1, \ldots, k$, and if $n = 2k + 1$ for some $k \geq 0$ then

$$S_n := \{-k, \ldots, -1, 0, 1, \ldots, k\},$$

with designated values $0, 1, \ldots, k$. For example, $S_1 := \{0\}$, $S_2 := \{-1, 1\}$, $S_3 := \{-1, 0, 1\}$, $S_4 := \{-2, -1, 1, 2\}$, and $S_5 := \{-2, -1, 0, 1, 2\}$. Note that $S_n \subseteq \mathbb{Z}$ so S_n is a chain under the natural ordering inherited from the ordering of the integers, that is,

$$-k < \cdots < -1 < 0 < 1 < \cdots < k.$$

With respect to the natural ordering of S_n the binary operations $\wedge$ and $\vee$ are defined as follows. For any $i, j \in S_n$, $i \wedge j$ is the minimum of i and j, and $i \vee j$ is the maximum of i and j. The unary operation $\sim$ is multiplication by -1, *i.e.*, it maps 0 to 0 (if n is odd and $0 \in S_n$), i to $-i$, and $-i$ to i whenever $0 < i \in S_n$. The binary operation $\to$ is defined for all $i, j \in S_n$ by

$$i \to j := \begin{cases} -i \vee j & \text{if } i \leq j \\ -i \wedge j & \text{if } i > j \end{cases} \tag{65}$$

The binary operation $\circ$, obtained by the definition $i \circ j := {\sim}(i \to {\sim}j)$, can be characterized as follows (Anderson and Belnap 1975, p. 400). If $1 \leq i, j \leq n$ then

$$-i \circ -j = -\max(i, j), \tag{66}$$

$$-i \circ j = \begin{cases} -i & \text{if } j \leq i \\ j & \text{if } i < j \end{cases}, \tag{67}$$

$$i \circ j = \max(i, j). \tag{68}$$

Said another way, $i \circ j$ is whichever of i and j is strictly larger in absolute value, or else is the minimum of i and j in case $|i| = |j|$. Another way to say this is that $i \circ j$ is the maximum of i and j under the linear ordering of S_n that begins in this way: $0 < 1 < -1 < 2 < -2 < 3 < -3 < 4 < -4 < \cdots$. Examples of $\circ$ are shown in Table 5.

The **Sugihara matrix** $\mathbf{S}_n$ is the algebra $\langle S_n, \vee, \wedge, \circ, \to, \sim\rangle$. $\mathbf{S}_n$ is **normal** if n is even. A sentence $A \in \mathsf{Sent}$ is **valid in** $\mathbf{S}_n$ if every homomorphism H from

$\circ$	-4	-3	-2	-1	1	2	3	4
-4	-4	-4	-4	-4	-4	-4	-4	-4
-3	-4	-3	-3	-3	-3	-3	-3	4
-2	-4	-3	-2	-2	-2	-2	3	4
-1	-4	-3	-2	-1	-1	2	3	4
1	-4	-3	-2	-1	1	2	3	4
2	-4	-3	-2	2	2	2	3	4
3	-4	-3	3	3	3	3	3	4
4	-4	4	4	4	4	4	4	4

$\circ$	-4	-3	-2	-1	0	1	2	3	4
-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
-3	-4	-3	-3	-3	-3	-3	-3	-3	4
-2	-4	-3	-2	-2	-2	-2	-2	3	4
-1	-4	-3	-2	-1	-1	-1	2	3	4
0	-4	-3	-2	-1	0	1	2	3	4
1	-4	-3	-2	-1	1	1	2	3	4
2	-4	-3	-2	2	2	2	2	3	4
3	-4	-3	3	3	3	3	3	3	4
4	-4	4	4	4	4	4	4	4	4

TABLE 5. Tables for $\circ$ in $\mathfrak{S}_8$ and $\mathfrak{S}_9$.

$\langle \mathsf{Sent}, \vee, \wedge, \circ, \rightarrow, \sim \rangle$ into $\mathbf{S}_n$ sends A to a designated value. Meyer's completeness theorem follows.

Theorem 6 (Meyer; see (Anderson and Belnap 1975, Cor. 3.1, p. 413))**.** *If a sentence A has no more than n propositional variables then $\vdash_{\mathbf{RM}} A$ iff A is valid in $\mathbf{S}_n$.*

We can now show that **RM** is complete for $\mathsf{R}^{\mathsf{cdt}}$.

Theorem 7. *Assume $1 \leq n < \omega$. Then there is a finite commutative dense transitive relational relevance algebra $\mathbf{T}_n \in \mathsf{R}^{\mathsf{cdt}}$ such that*

(i) *The Sugihara matrix $\mathbf{S}_{2n+2}$ is isomorphic to $\mathbf{T}_n$, so $\mathbf{S}_{2n+2} \in \underline{\mathbf{I}}\mathsf{R}^{\mathsf{cdt}}$.*
(ii) *If A has no more than $2n+2$ propositional variables, then $\vdash_{\mathbf{RM}} A$ iff $\mathbf{T}_n \models A$.*
(iii) *$\vdash_{\mathbf{RM}} A$ iff A is valid in $\mathsf{R}^{\mathsf{cdt}}$.*

Proof. Let $\mathbb{Q}^n := \{\langle q_1, \ldots, q_n \rangle : q_1, \ldots, q_n \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers. Define binary relations Id and L_1 on $\mathbb{Q}^n$ by

(69) $$\mathsf{Id} = \{\langle q, q \rangle : q \in \mathbb{Q}^n\},$$

(70) $$\langle q, q' \rangle \in L_1 \quad \text{iff} \quad q_1 < q'_1,$$

and for $1 < i \leq n$, define binary relations L_i on $\mathbb{Q}^n$ by

(71) $$\langle q, q' \rangle \in L_i \quad \text{iff} \quad \langle q_1, \ldots, q_{i-1} \rangle = \langle q'_1, \ldots, q'_{i-1} \rangle \text{ and } q_i < q'_i.$$

It follows that

$$\langle q, q' \rangle \in L_1 \cup L_1^{-1} \quad \text{iff} \quad q_1 \neq q'_1$$
$$\langle q, q' \rangle \in L_i \cup L_i^{-1} \quad \text{iff} \quad \langle q_1, \ldots, q_{i-1} \rangle = \langle q'_1, \ldots, q'_{i-1} \rangle \text{ and } q_i \neq q'_i.$$

Let

$$\mathcal{L}_n := \{\mathsf{Id}, L_1, L_1^{-1}, \dots, L_n, L_n^{-1}\}. \tag{72}$$

The relations in $\mathcal{L}_n$ are pairwise disjoint, and their union is $\mathbb{Q}^n \times \mathbb{Q}^n$. To see this, it is enough to observe that $\langle q, q' \rangle$ belongs to exactly one of the relations in $\mathcal{L}_n$. If $q = q'$, then for each $i = 1, \dots n$ it is not the case that $q_i < q'_i$ (hence $\langle q, q' \rangle \notin L_i$), nor is it the case that $q_i > q'_i$ (hence $\langle q, q' \rangle \notin L_i^{-1}$). Thus $\langle q, q' \rangle$ is not in any of the relations in $\{L_1, L_1^{-1}, \dots, L_n, L_n^{-1}\}$.

Assume $q \neq q'$. Let $i = 1$ if $q_1 \neq q'_1$, and otherwise let i be the smallest element of $\{2, \dots, n\}$ such that $\langle q_1, \dots, q_{i-1} \rangle = \langle q'_1, \dots, q'_{i-1} \rangle$ and $q_i \neq q'_i$. Since $\mathbb{Q}$ is linearly ordered, either $q_i < q'_i$ or $q_i > q'_i$, hence $\langle q, q' \rangle \in L_i$ iff $q_i < q'_i$ and $\langle q, q' \rangle \in L_i^{-1}$ iff $q_i > q'_i$. It follows from $\langle q_1, \dots, q_{i-1} \rangle = \langle q'_1, \dots, q'_{i-1} \rangle$ that $\langle q, q' \rangle$ is not in any of the relations L_1, L_1^{-1}, …, L_{i-1}, L_{i-1}^{-1}. The assumption that $q_i \neq q'_i$ prevents the pair from belonging to any of the remaining relations L_{i+1}, L_{i+1}^{-1}, …, L_n, L_n^{-1}.

Let

$$A_n := \{\bigcup S : S \subseteq \mathcal{L}_n\}. \tag{73}$$

Since the relations in $\mathcal{L}_n$ partition $\mathbb{Q}^n \times \mathbb{Q}^n$, A_n is the universe of a finite Boolean algebra of subsets of $\mathbb{Q}^n \times \mathbb{Q}^n$, and $\mathcal{L}_n$ is the set of atoms of this Boolean algebra. Clearly $\mathcal{L}_n$ is closed under conversion $^{-1}$, so A_n is also closed under $^{-1}$ because conversion distributes over union. Next we calculate the relative products of relations in $\mathcal{L}_n$.

Let $q, q'' \in \mathbb{Q}^n$. If $\langle q, q'' \rangle \in L_1|L_1$ then for some $q' \in \mathbb{Q}^n$ we have $q_1 < q'_1 < q''_1$, hence $\langle q, q'' \rangle \in L_1$. Conversely, if $\langle q, q'' \rangle \in L_1$ then we may choose $q' \in \mathbb{Q}^n$ so that $q'_1 = \frac{1}{2}(q_1 + q''_1)$, which yields $q_1 < q'_1 < q''_1$, hence $\langle q, q'' \rangle \in L_1|L_1$. Thus $L_1|L_1 = L_1$. If $1 < i \leq n$ and $\langle q, q'' \rangle \in L_i|L_i$, then there is some $q' \in \mathbb{Q}^n$ such that $\langle q, q' \rangle \in L_i$ and $\langle q', q'' \rangle \in L_i$, hence

$$\langle q_1, \dots, q_{i-1} \rangle = \langle q'_1, \dots, q'_{i-1} \rangle = \langle q''_1, \dots, q''_{i-1} \rangle \tag{74}$$

$$q_i < q'_i < q''_i, \tag{75}$$

so $\langle q, q'' \rangle \in L_i$. For the other direction, assume $\langle q, q'' \rangle \in L_i$. This gives us $\langle q_1, \dots, q_{i-1} \rangle = \langle q''_1, \dots, q''_{i-1} \rangle$ and $q_i < q''_i$, so we may choose $q' \in \mathbb{Q}^n$ such that (74) holds and $q'_i = \frac{1}{2}(q_i + q''_i)$, hence (75) also holds. We get $\langle q, q' \rangle \in L_i$ and $\langle q', q'' \rangle \in L_i$ from (74) and (75), hence $\langle q, q'' \rangle \in L_i|L_i$. So far we have proved

$$L_i|L_i = L_i \qquad \text{whenever } 1 \leq i \leq n. \tag{76}$$

Assume $1 < i < j \leq n$. If $\langle q, q'' \rangle \in L_i|(L_j \cup L_j^{-1})$, then there is some $q' \in \mathbb{Q}^n$ such that $\langle q, q' \rangle \in L_i$ and $\langle q', q'' \rangle \in L_j \cup L_j^{-1}$, hence

$$\langle q_1, \dots, q_{i-1} \rangle = \langle q'_1, \dots, q'_{i-1} \rangle \text{ and } q_i < q'_i$$
$$\langle q'_1, \dots, q'_{i-1}, q'_i, \dots, q'_{j-1} \rangle = \langle q''_1, \dots, q''_{i-1}, q''_i, \dots, q''_{j-1} \rangle \text{ and } q'_j \neq q''_j,$$

so

$$\langle q_1, \dots, q_{i-1} \rangle = \langle q''_1, \dots, q''_{i-1} \rangle \text{ and } q_i < q'_i = q''_i.$$

This proves that $L_i|(L_j \cup L_j^{-1}) \subseteq L_i$, hence $L_i|L_j \subseteq L_i$ and $L_i|L_j^{-1} \subseteq L_i$. To show the opposite inclusions, suppose $\langle q, q'' \rangle \in L_i$. Then

$$\langle q_1, \dots, q_{i-1} \rangle = \langle q''_1, \dots, q''_{i-1} \rangle \text{ and } q_i < q''_i.$$

If we let

$$q' = \left\langle q_1, \ldots, q_{i-1}, q''_i, \ldots, q''_{j-1}, q''_j - 1, q''_{j+1}, \ldots \right\rangle$$

then $\langle q, q' \rangle \in L_i$ and $\langle q', q'' \rangle \in L_j$, hence $\langle q, q'' \rangle \in L_i|L_j$, but if we let

$$q' = \left\langle q_1, \ldots, q_{i-1}, q''_i, \ldots, q''_{j-1}, q''_j + 1, q''_{j+1}, \ldots \right\rangle$$

then $\langle q, q' \rangle \in L_i$, $\langle q', q'' \rangle \in L_j^{-1}$, and $\langle q, q'' \rangle \in L_i|L_j^{-1}$. Except for the case $1 = i$, which is notationally simpler, we have completed the proof that

$$L_i = L_i|L_j = L_i|L_j^{-1} \qquad \text{whenever } 1 \leq i < j \leq n. \tag{77}$$

By very slightly rearranging the proof of (77) we also establish

$$L_i = L_j|L_i = L_j^{-1}|L_i \qquad \text{whenever } 1 \leq i < j \leq n. \tag{78}$$

By applying conversion to both sides of (76), (77), and (78) we also obtain

$$L_i^{-1} = L_i^{-1}|L_i^{-1} \qquad \text{whenever } 1 \leq i \leq n \tag{79}$$

$$L_i^{-1} = L_j^{-1}|L_i^{-1} = L_j|L_i^{-1} \qquad \text{whenever } 1 \leq i < j \leq n \tag{80}$$

$$L_i^{-1} = L_i^{-1}|L_j^{-1} = L_i^{-1}|L_j \qquad \text{whenever } 1 \leq i < j \leq n \tag{81}$$

Next we consider the products $L_i|L_i^{-1}$ and $L_i^{-1}|L_i$. If $\langle q, q'' \rangle \in L_i|L_i^{-1}$, then there is some $q' \in \mathbb{Q}^n$ such that $\langle q, q' \rangle \in L_i$ and $\langle q', q'' \rangle \in L_i^{-1}$, hence

$$\langle q_1, \ldots, q_{i-1} \rangle = \left\langle q'_1, \ldots, q'_{i-1} \right\rangle = \left\langle q''_1, \ldots, q''_{i-1} \right\rangle, \quad q_i < q'_i > q''_i.$$

There are three cases. First, if $q_i < q''_i$ then $\langle q, q'' \rangle \in L_i$. Second, if $q_i > q''_i$ then $\langle q, q'' \rangle \in L_i^{-1}$. For the third case we suppose $q_i = q''_i$, which implies

$$\langle q_1, \ldots, q_i \rangle = \langle q''_1, \ldots, q''_i \rangle. \tag{82}$$

If $q = q''$ then $\langle q, q'' \rangle \in \mathsf{Id}$. Suppose $q \neq q''$. From (82) we know that q and q'' must differ at some index greater than i. Let j be the smallest index such that $i < j \leq n$ and $q_j \neq q''_j$. If $q_j < q''_j$ then $\langle q, q'' \rangle \in L_j$. If $q_j > q''_j$ then $\langle q, q'' \rangle \in L_j^{-1}$. This exhausts all the possibilities, and shows that

$$L_i|L_i^{-1} \subseteq \mathsf{Id} \cup L_i \cup L_i^{-1} \cup \bigcup_{i<j\leq n} (L_j \cup L_j^{-1}) = \mathsf{Id} \cup \bigcup_{i\leq j\leq n} (L_j \cup L_j^{-1}).$$

For the opposite inclusion, assume

$$\langle q, q'' \rangle \in \mathsf{Id} \cup \bigcup_{i\leq j\leq n} (L_j \cup L_j^{-1}),$$

which is equivalent to

$$\langle q_1, \ldots, q_{i-1} \rangle = \left\langle q''_1, \ldots, q''_{i-1} \right\rangle. \tag{83}$$

If we choose $q' \in \mathbb{Q}^n$ so that

$$\langle q_1, \ldots, q_{i-1} \rangle = \left\langle q'_1, \ldots, q'_{i-1} \right\rangle = \left\langle q''_1, \ldots, q''_{i-1} \right\rangle \tag{84}$$

and $q'_i > \max(q_i, q''_i)$, we get $\langle q, q' \rangle \in L_i$ and $\langle q', q'' \rangle \in L_i^{-1}$, hence $\langle q, q'' \rangle \in L_i|L_i^{-1}$. This completes the proof that

$$L_i|L_i^{-1} = \mathsf{Id} \cup \bigcup_{i\leq j\leq n} (L_j \cup L_j^{-1}). \tag{85}$$

By slightly altering the proof of (85) we also get

$$L_i^{-1}|L_i = \mathsf{Id} \cup \bigcup_{i\le j\le n} (L_j \cup L_j^{-1}). \tag{86}$$

We can summarize (76)–(86) as follows.

$$L_i|L_j = L_{\min(i,j)}, \tag{87}$$

$$L_i^{-1}|L_j^{-1} = L_{\min(i,j)}^{-1}, \tag{88}$$

$$L_j^{-1}|L_i = L_i|L_j^{-1} = \begin{cases} L_i & \text{if } i<j \\ L_j^{-1} & \text{if } j<i \\ \mathsf{Id} \cup \bigcup_{i=j\le k\le n}(L_k \cup L_k^{-1}) & \text{if } i=j \end{cases}. \tag{89}$$

The remaining relative products of relations in $\mathcal{L}_n$, which all involve Id, are

$$\mathsf{Id} = \mathsf{Id}|\mathsf{Id}, \qquad L_i = L_i|\mathsf{Id} = \mathsf{Id}|L_i, \qquad L_i^{-1} = L_i^{-1}|\mathsf{Id} = \mathsf{Id}|L_i^{-1}. \tag{90}$$

Relative multiplication distributes over union, so it follows that A_n is closed under relative multiplication as well as union, intersection, complementation with respect to $\mathbb{Q}^n \times \mathbb{Q}^n$, and conversion. Note that A_n contains the identity relation on $\mathbb{Q}^n$.

For every $J \subseteq \{1,2,\ldots,n\}$, let $L_J := \emptyset$ and $L_J^{-1} := \emptyset$ if $J=\emptyset$, and otherwise let $L_J := \bigcup_{i\in J} L_i$ and $L_J^{-1} := \bigcup_{i\in J} L_i^{-1}$. For every $i \in \{1,2,\ldots,n\}$ let $[1,i] = \{1,2,\ldots,i-1,i\}$ and $[i,n] = \{i,i+1,\ldots,n-1,n\}$. Using this notation we can rewrite (85) and (86) as

$$L_i^{-1}|L_i = L_i|L_i^{-1} = \mathsf{Id} \cup L_{[i,n]} \cup L_{[i,n]}^{-1}, \tag{91}$$

and derive a few more computational rules.

$$L_{[1,i]}|L_{[1,j]} = \bigcup_{1\le k\le i,\, 1\le l\le j} L_k|L_l = \bigcup_{1\le k\le i,\, 1\le l\le j} L_{\min(k,l)} = L_{[1,\min(i,j)]}, \tag{92}$$

$$L_{[i,n]}|L_{[j,n]} = \bigcup_{i\le k\le n,\, j\le l\le n} L_k|L_l = \bigcup_{i\le k\le n,\, j\le l\le n} L_{\min(k,l)} = L_{[\min(i,j),n]}, \tag{93}$$

$$L_{[1,i]}^{-1}|L_{[1,j]}^{-1} = L_{[1,\min(i,j)]}^{-1}, \tag{94}$$

$$L_{[i,n]}^{-1}|L_{[j,n]}^{-1} = L_{[\min(i,j),n]}^{-1}. \tag{95}$$

If $i<j$ then $L_k|L_l^{-1} = L_k$ whenever $1\le k\le i$ and $j\le l\le n$, so

$$L_{[1,i]}|L_{[j,n]}^{-1} = \bigcup_{1\le k\le i,\, j\le l\le n} L_k|L_l^{-1} = \bigcup_{1\le k\le i,\, j\le l\le n} L_k = L_{[1,i]}. \tag{96}$$

On the other hand, if $1\le j\le i$ then

$$\begin{aligned} L_{[1,i]}|L_j^{-1} &= \bigcup_{1\le k<j} L_k|L_j^{-1} \cup L_j|L_j^{-1} \cup \bigcup_{j<k\le i} L_k|L_j^{-1} \\ &= \bigcup_{1\le k<j} L_k \cup \mathsf{Id} \cup L_{[j,n]} \cup L_{[j,n]}^{-1} \cup \bigcup_{j<k\le i} L_j^{-1} \\ &= L_{[1,j-1]} \cup \mathsf{Id} \cup L_{[j,n]} \cup L_{[j,n]}^{-1} \cup L_j^{-1} \\ &= L_{[1,n]} \cup \mathsf{Id} \cup L_{[j,n]}^{-1}, \end{aligned}$$

which implies

(97) $L_{[1,i]}|L^{-1}_{[j,n]} = L_{[1,n]} \cup \mathsf{Id} \cup L^{-1}_{[j,n]}$ whenever $1 \le j \le i$.

We will use the relations in $\mathcal{L}_n$ to create a copy of the Sugihara matrix $\mathbf{S}_{2n+2}$. The example which inspired this construction is Belnap's $\mathbf{M}_0$, which has two copies of $\mathbf{S}_4$ as subalgebras, namely $\{-3, -2, +2, +3\}$ and $\{-3, -1, +1, +3\}$.

First, define a function $T: S_{2n+2} \to Sb\left(\mathbb{Q}^{n^2}\right)$ by

(98) $T_{-n-1} := \emptyset$

(99) $T_{-i} := L_{[1,n+1-i]}$ whenever $1 \le i \le n$

(100) $T_1 := L_{[1,n]} \cup \mathsf{Id}$

(101) $T_i := L_{[1,n]} \cup \mathsf{Id} \cup L^{-1}_{[n+2-i,n]}$ whenever $2 \le i \le n+1$

and let

$$\mathcal{T}_n := \{T_{-n-1}, T_{-n}, \ldots, T_{-1}, T_1, \ldots, T_n, T_{n+1}\}.$$

Note that $T_{n+1} = \mathbb{Q}^n \times \mathbb{Q}^n$. Also, the images of the designated values of $\mathbf{S}_{2n+2}$ are $T_1, \ldots, T_n, T_{n+1}$, exactly the elements of $\mathcal{T}_n$ that contain the identity relation Id. It follows immediately from the definitions that the relations in $\mathcal{T}_n$ form a chain,

(102) $$T_{-n-1} \subseteq T_{-n} \subseteq \cdots \subseteq T_{-1} \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq T_{n+1}.$$

Therefore $\mathcal{T}_n$ is closed under union and intersection. A straightforward calculation shows that $\mathcal{T}_n$ is also closed under converse-complementation. In fact, for every $i \in \{-n-1, \ldots, -1, 1, \ldots, n+1\} = S_{2n+2}$ we have

(103) $$\sim(T_i) = T_{-i} = T_{\sim(i)}.$$

To show that $\mathcal{T}_n$ is closed under relative multiplication, we need to examine all the products of relations in $\mathcal{T}_n$.

First note that all products involving $T_{-n-1} = \emptyset$ are pretty trivial, for if $X \in T_n$ then

(104) $$T_{-n-1}|X = \emptyset|X = \emptyset = T_{-n-1}.$$

If $1 \le i, j \le n$ then we have

(105) $$T_{-i}|T_{-j} = T_{-\max(i,j)},$$

since

$$\begin{aligned} T_{-i}|T_{-j} &= L_{[1,n+1-i]}|L_{[1,n+1-j]} \\ &= L_{[1,\min(n+1-i,n+1-j)} && \text{by (92)} \\ &= L_{[1,n+1-\max(i,j)]} \\ &= T_{-\max(i,j)}. \end{aligned}$$

We use this to show

(106) $$T_{-i}|T_1 = T_{-i}$$

as follows.

$$\begin{aligned} T_{-i}|T_1 &= T_{-i}|(T_{-1} \cup \mathsf{Id}) && \text{by (100)} \\ &= T_{-i}|T_{-1} \cup T_{-i}|\mathsf{Id} \\ &= T_{-\max(i,1)} \cup T_{-i} && \text{by (105) with } j = 1 \\ &= T_{-i} \cup T_{-i} \end{aligned}$$

$$= T_{-i}.$$

If $n \geq i \geq j \geq 1$, then $n+1-i < n+2-j$, so by (96),

$$T_{-i}|L^{-1}_{[n+2-j,n]} = L_{[1,n+1-i]}|L^{-1}_{[n+2-j,n]} = L_{[1,n+1-i]} = T_{-i}.$$

Using these last two observations we get

$$\begin{aligned} T_{-i}|T_j &= T_{-i}|(T_1 \cup L^{-1}_{[n+2-j,n]}) \\ &= T_{-i}|T_1 \cup T_{-i}|L^{-1}_{[n+2-j,n]} \\ &= T_{-i} \cup T_{-i} = T_{-i}. \end{aligned}$$

On the other hand, if $1 \leq i < j \leq n$ then $n+1-i \geq n+2-j$, so by (97)

$$T_{-i}|L^{-1}_{[n+2-j,n]} = L_{[1,n+1-i]}|L^{-1}_{[n+2-j,n]} = L_{[1,n]} \cup \mathsf{Id} \cup L^{-1}_{[n+2-j,n]} = T_j,$$

hence

$$\begin{aligned} T_{-i}|T_j &= T_{-i}|(T_1 \cup L^{-1}_{[n+2-j,n]}) \\ &= T_{-i}|T_1 \cup T_{-i}|L^{-1}_{[n+2-j,n]} \\ &= T_{-i} \cup T_j = T_j. \end{aligned}$$

We have proved that

$$T_{-i}|T_j = \begin{cases} T_{-i} & \text{if } n \geq i \geq j \geq 1 \\ T_j & \text{if } 1 \leq i < j \leq n \end{cases}. \tag{107}$$

Next we deal with one special product.

$$\begin{aligned} T_1|T_1 &= (L_{[1,n]} \cup \mathsf{Id})|(L_{[1,n]} \cup \mathsf{Id}) \\ &= L_{[1,n]}|L_{[1,n]} \cup \mathsf{Id}|L_{[1,n]} \cup L_{[1,n]}|\mathsf{Id} \cup \mathsf{Id}|\mathsf{Id} \\ &= L_{[1,n]} \cup L_{[1,n]} \cup L_{[1,n]} \cup \mathsf{Id} && \text{by (92)} \\ &= T_1. \end{aligned} \tag{108}$$

Suppose $2 \leq j \leq n$. First observe that

$$\begin{aligned} T_1|L^{-1}_{[n+2-j,n]} &= (L_{[1,n]} \cup \mathsf{Id})|L^{-1}_{[n+2-j,n]} \\ &= L_{[1,n]}|L^{-1}_{[n+2-j,n]} \cup \mathsf{Id}|L^{-1}_{[n+2-j,n]} \\ &= L_{[1,n]} \cup \mathsf{Id} \cup L^{-1}_{[n+2-j,n]} \cup L^{-1}_{[n+2-j,n]} && \text{by (97)} \\ &= T_j, \end{aligned} \tag{109}$$

and then use this observation together with (108) to obtain

$$\begin{aligned} T_1|T_j &= T_1|\left(T_1 \cup L^{-1}_{[n+2-j,n]}\right) \\ &= T_1|T_1 \cup T_1|L^{-1}_{[n+2-j,n]} \\ &= T_1 \cup T_j = T_j && \text{by (108), (109).} \end{aligned} \tag{110}$$

Finally, if $2 \leq i, j \leq n+1$ then we first note

$$\begin{aligned} T_i|L^{-1}_{[n+2-j,n]} &= \left(T_1 \cup L^{-1}_{[n+2-i,n]}\right)|L^{-1}_{[n+2-j,n]} \\ &= T_1|L^{-1}_{[n+2-j,n]} \cup L^{-1}_{[n+2-i,n]}|L^{-1}_{[n+2-j,n]} \\ &= T_j \cup L^{-1}_{[\min(n+2-i,n+2-j),n]} && \text{by (109), (95)} \end{aligned} \tag{111}$$

$$= T_j \cup L^{-1}_{[n+2-\max(i,j),n]},$$

and then

$$(112) \qquad \begin{aligned} T_i|T_j &= T_i|\left(T_1 \cup L^{-1}_{[n+2-j,n]}\right) \\ &= T_i|T_1 \cup T_i|L^{-1}_{[n+2-j,n]} \\ &= T_i \cup T_j \cup L^{-1}_{[n+2-\max(i,j),n]} && \text{by (110), (111)} \\ &= T_{\max(i,j)} \cup L^{-1}_{[n+2-\max(i,j),n]} \\ &= T_{\max(i,j)}. \end{aligned}$$

This completes the proof that $\mathcal{T}_n$ is closed under relative multiplication and composition. Since $\mathcal{T}_n$ is closed under $\cup$, $\cap$, $\circ$, $\rightarrow$, $\sim$, we may use it as the universe of an algebra with these operations. Let

$$(113) \qquad \mathbf{T}_n := \langle \mathcal{T}_n, \cup, \cap, \circ, \rightarrow, \sim \rangle .$$

Observe that (105), (107), (108), (110), and (112) are enough to confirm that relative multiplication in $\mathbf{T}_n$ behaves the same as multiplication in the Sugihara matrix $\mathbf{S}_{2n+2}$ according to (66)–(68). It is easy to see that the other operations are preserved by T, so T is an isomorphism from the Sugihara matrix $\mathbf{S}_{2n+2}$ to $\mathbf{T}_n$. Combining these observations with Theorem 6 completes the proof of part (i).

For part (ii), consider a sentence A and choose n so that A has fewer than $2n+2$ propositional variables. By Theorem 6 we have $\vdash_{\mathbf{RM}} A$ iff A is valid in $\mathbf{S}_{2n+2}$. The isomorphism from $\mathbf{S}_{2n+2}$ to $\mathbf{T}_n$ carries designated values of $\mathbf{S}_{2n+2}$ onto the relations in $\mathbf{T}_n$ that contain Id, so A is valid in $\mathbf{S}_{2n+2}$ iff $\mathbf{T}_n \models A$. Part (iii) follows from parts (i) and (ii). □

7. Relevant model structures

Relevant model structures, introduced in (Routley and Meyer 1973; Routley and Meyer 1972a; Routley and Meyer 1972b), provide sound and complete semantics for **R**. A **relevant model structure** $\mathfrak{K} = \langle K, R, {}^*, 0\rangle$ consists of a non-empty set K, a ternary relation $R \subseteq K^3$, a unary operation ${}^* : K \to K$, and a distinguished element $0 \in K$, such that postulates (p1)–(p6) hold for all $a, b, c \in K$. To state these postulates, we first adopt some definitions.

(d1) $\quad R^2abcd \quad \text{iff} \quad \exists_x(Rabx, Rxcd, x \in K)$

(d2) $\quad R^2a(bc)d \quad \text{iff} \quad \exists_x(Rbcx, Raxd, x \in K)$

(d3) $\quad b \leq_a c \quad \text{iff} \quad Rabc$

The defining properties of relevant model structures are

(p1) $\quad R0aa$ (0-reflexivity)

(p2) $\quad Raaa$ (density)

(p3) $\quad R^2abcd \implies R^2acdb$

(p4) $\quad R^20abc \implies Rabc$ (0-cancellation)

(p5) $\quad Rabc \implies Rac^*b^*$

(p6) $\quad a^{**} = a$ (involution)

Next are four more properties of relevant model structures, as shown in Theorem 8 below.

(comm)	$Rabc \implies Rbac$	(commutativity)
(p3$'$)	$R^2abcd \implies R^2a(bc)d$	(associativity)
(p5$'$)	$Rabc \implies Rc^*ab^*$	(right rotation)
(p5$''$)	$Rabc \implies Rbc^*a^*$	(left rotation)

The next three properties do not hold in all relevant model structures.

(p1$'$)	$R0ab \quad \text{iff} \quad a = b$	(0-identity)
(p5$'''$)	$Rabc \implies Rcb^*a$	(right reflection)
(p5$''''$)	$Rabc \implies Ra^*cb$	(left reflection)

The rotation properties (p5$'$) and (p5$''$) are equivalent in the presence of (p6).

The **reflections** of a triple $\langle a,b,c\rangle$ are $\langle c,b^*,a\rangle$, $\langle a^*,c,b\rangle$, and $\langle b^*,a^*,c^*\rangle$. The **rotations** of a triple $\langle a,b,c\rangle$ are $\langle a,b,c\rangle$, $\langle c^*,a,b^*\rangle$, and $\langle b,c^*,a^*\rangle$. The ternary relation $[a,b,c]$ defined by

$$[a,b,c] := \{\langle a,b,c\rangle, \langle c^*,a,b^*\rangle, \langle b,c^*,a^*\rangle, \langle a^*,c,b\rangle, \langle c,b^*,a\rangle, \langle b^*,a^*,c^*\rangle\} \tag{114}$$

is called a **cycle**. It is the closure of $\{\langle a,b,c\rangle\}$ under (both left and right) rotations and reflections. Any union of cycles will satisfy both rotation and reflection properties. The size of a cycle is 1, 2, 3, or 6, depending on the behovior of * on a, b, and c.

A relevant model structure $\mathfrak{K} = \langle K, R, {}^*, 0\rangle$ is **normal** if $0^* = 0$. If a relevant model structure $\mathfrak{K}$ satisfies (p1$'$) then $\mathfrak{K}$ is normal, because $R0^*0^*0^*$ by (p2), $R0^*00$ by (p5) and involution (p6), $R00^*0$ by (comm), so $0^* = 0$ by (p1$'$).

Theorem 8. *Properties* (p1)–(p6) *are equivalent to* (p1)*,* (p2)*,* (p3$'$)*,* (p4)*,* (p5$'$) (p6)*, and* (comm)*.*

Proof. Assume postulates (p1)–(p6). We must show (comm), (p3$'$), and (p5$'$). For this we only need (p3), (p4), and (p5).

Assume $Rabc$. We have $R0aa$ by (p1), so R^20abc by (d2), hence R^20bac by (p3), and finally $Rbac$ by (p4). Thus (comm) holds. (p5$'$) follows from (p5) by (comm). For (p3$'$), assume R^2abcd. Then R^2bacd by (d2) and (comm), so R^2bcad by (p3), and finally $R^2a(bc)d$ by (d1), (comm), and (d2).

For the converse, assume (p1), (p2), (p3$'$), (p4), (p5$'$), (p6), and (comm). We get (p5) from (p5$'$) and (comm). For (p3), assume R^2abcd. Then R^2bacd by (d1) and (comm), hence $R^2b(ac)d$ by (p3$'$), and finally R^2acbd by (d2), (comm), and (d1). □

Because of this theorem we think of a relevant model structure as one that satisfies 0-reflexivity, 0-cancellation, density, involution, associativity, commutativity, and both rotations.

Suppose $\mathfrak{K} = \langle K, R, {}^*, 0\rangle$ is a structure with distinguished element $0 \in K$, ternary relation $R \subseteq K^3$, and unary operation $^* : K \to K$. ($\mathfrak{K}$ need not be a relevant model structure). For any $a \in K$ and $X \subseteq K$, X is ***a*-closed** if $y \in X$ whenever $x \in X$ and $x \leq_a y$. Let $\Pi(\mathfrak{K})$ be the set of 0-closed subsets of $\mathfrak{K}$. A **valuation** on $\mathfrak{K}$ is a function $\nu : \mathsf{Sent} \to Sb(K)$ such that, for all $A, B \in \mathsf{Sent}$,

$$\nu(A) \in \Pi(\mathfrak{K}) \text{ if } A \in \mathsf{Pv},$$

$$\nu(A \wedge B) = \nu(A) \cap \nu(B),$$
$$\nu(A \vee B) = \nu(A) \cup \nu(B),$$
$$\nu(A \circ B) = \{c : (\exists a, b \in K)(Rabc \text{ and } a \in \nu(A) \text{ and } b \in \nu(B))\},$$
$$\nu(A \to B) = \{c : (\forall a, b \in K)(\text{if } Rcab \text{ and } a \in \nu(A) \text{ then } b \in \nu(B))\},$$
$$\nu(\sim A) = \{a : a^* \notin \nu(A)\}.$$

We say that $A \in \mathsf{Sent}$ is **valid in** $\mathfrak{K}$ if $0 \in \nu(A)$ for every valuation ν on $\mathfrak{K}$.

Define operations $\circ$, $\to$, and $\sim$ on subsets $X, Y \subseteq K$ by

(115) $$X \circ Y = \{c : (\exists a, b \in K)(Rabc \text{ and } a \in X \text{ and } b \in Y)\}$$

(116) $$X \to Y := \{c : (\forall a, b \in K)(\text{if } Rcab \text{ and } a \in X \text{ then } b \in Y)\},$$

(117) $$\sim X := \{a : a^* \notin X\}.$$

If $\mathfrak{K}$ satisfies (p6) and (p5′) then $X \circ Y = \sim(X \to \sim Y)$ and $X \to Y = \sim(X \circ \sim Y)$. The next lemma shows that the set of 0-closed subsets of a relevant model structure is closed under union, intersection, and the operations $\circ$, $\to$, and $\sim$.

Lemma 9 (Routley and Meyer 1973, Lem. 1)**.** *If $\mathfrak{K}$ is a relevant model structure, ν is a valuation on $\mathfrak{K}$, and $A \in \mathsf{Sent}$, then $\nu(A) \in \Pi(\mathfrak{K})$.*

Since $\Pi(\mathfrak{K})$ is closed under the operations $\cup$, $\cap$, $\circ$, $\to$, and $\sim$, we define the **algebra of** $\mathfrak{K}$ to be

(118) $$\mathfrak{Pr}(\mathfrak{K}) := \langle \Pi(\mathfrak{K}), \cup, \cap, \circ, \to, \sim \rangle$$

In this definition we avoid distinguished elements, but they are sometimes included; see (Routley and Meyer 1973, p. 228) and (Brady 2003, p. 81) for other choices of similarity type for the algebra of $\mathfrak{K}$.

The algebra $\mathfrak{Pr}(\mathfrak{K})$ of a relevant model structure $\mathfrak{K}$ is a subalgebra of a larger algebra obtained by using the set of all subsets of K instead of $\Pi(\mathfrak{K})$. This is the **complex algebra** of $\mathfrak{K}$, defined by

(119) $$\mathfrak{Cm}(\mathfrak{K}) := \langle Sb(K), \cup, \cap, \circ, \to, \sim \rangle .$$

Note that if 0-identity property (p1′) holds in $\mathfrak{K}$, then $\mathfrak{Cm}(\mathfrak{K})$ coincides with the algebra of $\mathfrak{K}$.

Furthermore, the complex algebra $\mathfrak{Cm}(\mathfrak{K})$ has no $\sim$-fixed points. To see this, suppose $X = \sim X = \{a : a^* \notin X\}$ for some $X \subseteq K$. Then $a \in X \quad \text{iff} \quad a^* \notin X$, for all $a \in K$. In particular, for $a = 0$ we would have $0 \in X \quad \text{iff} \quad 0^* \notin X$, but $0 = 0^*$ in every relevant model structure satisfying (p1′), a contradiction.

Every valuation ν on a relevant model structure $\mathfrak{K}$ is a homomorphism from the algebra of sentences $\langle \mathsf{Sent}, \vee, \wedge, \circ, \to, \sim \rangle$ to the algebra of $\mathfrak{K}$, and conversely. Therefore A is valid in $\mathfrak{K}$ iff $0 \in \nu(A)$ for every homomorphism ν from $\langle \mathsf{Sent}, \vee, \wedge, \circ, \to, \sim \rangle$ to the algebra of $\mathfrak{K}$.

The following two constructions are from (Meyer and Routley 1973, Part I) and (Routley and Meyer 1973). For both of them we assume $\mathfrak{K} = \langle K, R, {}^*, 0 \rangle$ where $R \subseteq K^3$, ${}^* : K \to K$, and $0 \in K$. Let $0' \notin K$ and let $K' := K \cup \{0'\}$. Define a unary operation ${}^{*'} : K' \to K'$ as follows: $a^{*'} = a^*$ if $a \in K$ and $0'^{*'} = 0'$. Let R' be the ternary relation on K' defined by

$$\begin{aligned} R' :=\, & R \cup \{\langle 0', 0', 0' \rangle\} \\ & \cup \{\langle 0', 0', a \rangle : \langle 0, 0, a \rangle \in R\} \end{aligned}$$

$$\cup\{\langle 0',a,0'\rangle:\langle 0,a,0^*\rangle\in R\}$$
$$\cup\{\langle a,0',0'\rangle:\langle a,0,0^*\rangle\in R\}$$
$$\cup\{\langle a,b,0'\rangle:\langle a,b,0^*\rangle\in R\}$$
$$\cup\{\langle 0',a,b\rangle:\langle 0,a,b\rangle\in R\}$$
$$\cup\{\langle a,0',b\rangle:\langle a,0,b\rangle\in R\},$$

and let $\mathfrak{K} := \left\langle K', R', {}^{*'}, 0' \right\rangle$. Then $\mathfrak{K}'$ is the **normalization** of $\mathfrak{K}$.

Lemma 10 (Routley and Meyer 1973). *If $\mathfrak{K}$ is a relevant model structure then the normalization of $\mathfrak{K}$ is a normal relevant model structure. If a sentence $A \in \mathsf{Sent}$ is invalid in $\mathfrak{K}$, then A is also invalid in the normalization of $\mathfrak{K}$.*

For a similar construction from (Meyer and Routley 1973, Part I), choose some 1' $\notin K$ and let $K' := K \cup \{1'\}$. Define a unary operation ${}^{*'} : K' \to K'$ as follows: $a^{*'} = a^*$ if $a \in K$ and $1'^{*'} = 1'$. Define a ternary relation

$$R' := R \cup \{\langle a,1',a\rangle : a\in K\} \cup \{\langle 1',a,a\rangle : a\in K\}$$
$$\cup\{\langle a,a^*,1'\rangle : a\in K\}\cup\{\langle 1',1',1'\rangle\}$$

and let $\mathfrak{K}' := \left\langle K', R', {}^{*'}, 1' \right\rangle$. Meyer and Routley (1973, Part I) did not give a name to $\mathfrak{K}'$. We will call it **$\mathfrak{K}$-with-identity**, and denote it briefly by $\mathfrak{K}[1']$.

Lemma 11 (Meyer and Routley 1973, Part I). *If $\mathfrak{K}$ is a relevant model structure then $\mathfrak{K}[1']$ is a normal relevant model structure that satisfies* (p1′)*. Furthermore, if $\mathfrak{K}$ is normal then exactly the same sentences are valid in both $\mathfrak{K}$ and $\mathfrak{K}[1']$.*

Next are the Routley-Meyer completeness results.

Theorem 12 (Routley and Meyer 1973, Meyer and Routley 1973). *The following statements are equivalent for every sentence $A \in \mathsf{Sent}$.*
- (i) $\vdash_{\mathbf{R}} A$.
- (ii) *A is valid in every relevant model structure.*
- (iii) *A is valid in every normal relevant model structure.*
- (iv) *A is valid in every relevant model structure that satisfies* (p1′).

Proof. The equivalence of (i) and (ii) is Theorem 3 of (Routley and Meyer 1973). Obviously (ii) implies (iii), and (iii) implies (iv) since every relevant model structure that satisfies (p1′) is normal.

To show that (iv) implies (i) it is enough to prove that every non-theorem of $\mathbf{R}$ is invalid in some (normal) relevant model structure that satisfies (p1′). Assume $\nvdash_{\mathbf{R}} A$. Since (ii) implies (i), there exists some relevant model structure $\mathfrak{K}$ such that A is not valid in $\mathfrak{K}$. Let $\mathfrak{K}'$ be the normalization of $\mathfrak{K}$ and let $\mathfrak{K}''$ be $\mathfrak{K}'[1']$. Thus $\mathfrak{K}''$ has two more elements than $\mathfrak{K}$. Since A is invalid in $\mathfrak{K}$, it is also invalid in the normalization $\mathfrak{K}'$ of $\mathfrak{K}$ by Lemma 10. But the same sentences are valid in both $\mathfrak{K}'$ and $\mathfrak{K}''$ by Lemma 11, so A is also invalid in $\mathfrak{K}''$. Since $\mathfrak{K}''$ is a relevant model structure that satisfies property (p1′), we are done. □

Part (iv) of Theorem 12 inspired the following question, which was asked in (Maddux 2007).

(Q3) Is is true that $\vdash_{\mathbf{R}} A$ iff A is valid in every relevant model structure that satisfies (p1′), (p5′′′), and (5′′′′)?

In addressing this question, Kowalski (2007) defines a system **B** whose language contains only $\wedge$, $\vee$, and $\to$. The axioms of **B** are (A1)–(A8) and the rules are *modus ponens*, Adjunction, Prefixing, and Suffixing. He proves that $\vdash_{\mathbf{B}} A$ iff A is valid in every structure that satisfies (p6), (p1′), (p5‴), (5⁗), plus the condition that $Ra0b$ iff $a = b$.

8. Incompleteness of R for R^{cd}

We answer question (Q1) here, for which we will need

Theorem 13. *Let U be a non-empty set and assume $A, B, C, D, E, F, G \subseteq U^2$. Then*

$$\text{(L)}\qquad \mathsf{Id} \subseteq A|B \cap C|D \cap E|F \to A|\Big(A^{-1}|C \cap B|D^{-1} \cap (A^{-1}|E \cap B|F^{-1})|(E^{-1}|C \cap F|D^{-1})\Big)|D,$$

$$\text{(L}'\text{)}\qquad \mathsf{Id} \subseteq A|B \cap C|D \cap E|F \to \Big((A \cap {\sim}A)|B \cap C|D \cap E|F\Big) \cup \Big(A|B \cap C|(D \cap {\sim}D) \cap E|F\Big) \cup \Big(A|B \cap C|D \cap (E \cap {\sim}E)|F\Big) \cup \Big(A|B \cap C|D \cap E|(F \cap {\sim}F)\Big) \cup A|\Big(A|C \cap B|D \cap (A|E \cap B|F)|(E|C \cap F|D)\Big)|D,$$

$$\text{(M)}\qquad \mathsf{Id} \subseteq A \cap (B \cap C|D)|(E \cap F|G) \to C|\Big((C^{-1}|A \cap D|E)|G^{-1} \cap D|F \cap C^{-1}|(A|G^{-1} \cap B|F)\Big)|G,$$

$$\text{(M}'\text{)}\qquad \mathsf{Id} \subseteq A \cap (B \cap C|D)|(E \cap F|G) \to \Big(A \cap (B \cap (C \cap {\sim}C)|D)|(E \cap F|G)\Big) \cup \Big(A \cap (B \cap C|D)|(E \cap F|(G \cap {\sim}G))\Big) \cup C|\Big((C|A \cap D|E)|G \cap D|F \cap C|(A|G \cap B|F)\Big)|G.$$

Parts (L) and (M) are in the calculus of relations, but they are not part of relevance logic because they involve conversion. Accompanying (L) and (M) are their consequences (L′) and (M′). These use only the operations allowed in relevance logic but, as is shown below, their corresponding sentences are not provable in **R**.

Infinitely many more such examples can be found in (Mikulás 2008). Now (L), (M), and the equations used by Mikulás (2008) all have the same special form. There is a general procedure applicable to such equations which produces (L′) and (M′) from (L) and (M), respectively. There are also procedures that work on all equations if a particular constant is available in the language. However, we will not go further into these matters.

Proof of Theorem 13. We only prove (M) and (M′). The proofs of (L) and (L′) are similar. By (17), (M) and (M′) are equivalent to inclusions whose left side is the relation $A \cap (B \cap C|D)|(E \cap F|G)$.

For (M), suppose $\langle v, w\rangle \in A \cap (B \cap C|D)|(E \cap F|G)$. Then $\langle v, w\rangle \in A$ and there is some $x \in U$ such that $\langle v, x\rangle \in B$, $\langle v, x\rangle \in C|D$, $\langle x, w\rangle \in E$, and $\langle x, w\rangle \in F|G$. Hence there are $y, z \in U$ such that $\langle v, y\rangle \in C$, $\langle y, x\rangle \in D$, $\langle x, z\rangle \in F$, and $\langle z, w\rangle \in G$. It now follows from only $\langle v, w\rangle \in A$, $\langle v, x\rangle \in B$, $\langle x, w\rangle \in E$, $\langle v, y\rangle \in C$, $\langle y, x\rangle \in D$,

$\langle x,z\rangle \in F$, and $\langle z,w\rangle \in G$ that $\langle v,w\rangle$ is in the relation in the conclusion of (M), that is,

$$\langle v,w\rangle \in C|\Big((C^{-1}|A\cap D|E)|G^{-1}\cap D|F\cap C^{-1}|(A|G^{-1}\cap B|F)\Big)|G.$$

For (M′), suppose $\langle v,w\rangle \in A\cap(B\cap C|D)|(E\cap F|G)$. Then, as before, there are $x,y,z\in U$ such that $\langle v,w\rangle\in A$, $\langle v,x\rangle\in B$, $\langle x,w\rangle\in E$, $\langle v,y\rangle\in C$, $\langle y,x\rangle\in D$, $\langle x,z\rangle\in F$, and $\langle z,w\rangle\in G$. If $\langle y,v\rangle\notin C$ or $\langle w,z\rangle\notin G$, then $\langle v,y\rangle\in C\cap{\sim}C$ or $\langle z,w\rangle\in G\cap{\sim}G$, respectively, and in either case $\langle v,w\rangle$ belongs to one of the first two relations in the conclusion of (M′). Hence

$$\langle v,w\rangle \in \big(A\cap(B\cap(C\cap{\sim}C)|D)|(E\cap F|G)\big)\cup\big(A\cap(B\cap C|D)|(E\cap F|(G\cap{\sim}G))\big).$$

On the other hand, if $\langle y,v\rangle\in C$ and $\langle w,z\rangle\in G$, then $\langle v,y\rangle\in C\cap C^{-1}$ and $\langle z,w\rangle\in G\cap G^{-1}$, so

$$\langle v,w\rangle\in A\cap(B\cap(C\cap C^{-1})|D)|(E\cap F|(G\cap G^{-1})).$$

Now apply (M) with $C\cap C^{-1}$ and $G\cap G^{-1}$ in place of C and G, respectively, and conclude that $\langle v,w\rangle$ belongs to a relation contained in the third relation in the conclusion of (M′), as follows.

$$\begin{aligned}\langle v,w\rangle &\in (C\cap C^{-1})|\Big(((C\cap C^{-1})^{-1}|A\cap D|E)|(G\cap G^{-1})^{-1}\\ &\qquad \cap D|F\cap(C\cap C^{-1})^{-1}|(A|(G\cap G^{-1})^{-1}\cap B|F)\Big)|G\\ &= (C\cap C^{-1})|\Big(((C\cap C^{-1})|A\cap D|E)|(G\cap G^{-1})\\ &\qquad \cap D|F\cap(C\cap C^{-1})|(A|(G\cap G^{-1})\cap B|F)\Big)|G\\ &\subseteq C|\Big((C|A\cap D|E)|G\cap D|F\cap C|(A|G\cap B|F)\Big)|G.\end{aligned}$$

□

We use the abbreviation $A|B := {\sim}(B\to{\sim}A)$ to transcribe (L′) and (M′) into sentences (L″), (M″) $\in$ Sent.

(M″)
$$\begin{aligned}&A\wedge(B\wedge C|D)|(E\wedge F|G)\to\\ &\Big(A\wedge(B\wedge(C\wedge{\sim}C)|D)|(E\wedge F|G)\Big)\\ \vee&\Big(A\wedge(B\wedge C|D)|(E\wedge F|(G\wedge{\sim}G))\Big)\\ \vee&\, C|\Big((C|A\wedge D|E)|G\wedge D|F\wedge C|(A|G\wedge B|F)\Big)|G\end{aligned}$$

(L″)
$$\begin{aligned}&A|B\wedge C|D\wedge E|F\to\\ &\Big((A\wedge{\sim}A)|B\wedge C|D\wedge E|F\Big)\vee\Big(A|B\wedge C|(D\wedge{\sim}D)\wedge E|F\Big)\\ \vee&\Big(A|B\wedge C|D\wedge(E\wedge{\sim}E)|F\Big)\vee\Big(A|B\wedge C|D\wedge E|(F\wedge{\sim}F)\Big)\\ \vee&\, A|\Big(A|C\wedge B|D\wedge(A|E\wedge B|F)|(E|C\wedge F|D)\Big)|D\end{aligned}$$

The validity of (L″) and (M″) in R was established by Theorem 13. However,

Theorem 14. $\nvdash_{\mathbf{R}}$ (L″) *and* $\nvdash_{\mathbf{R}}$ (M″).

$\circ$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
$\{1\}$	$\{1\}$	$\{0,1,3\}$	$\{2,3\}$	$\{1,2\}$
$\{2\}$	$\{2\}$	$\{2,3\}$	$\{0,1,2\}$	$\{1,3\}$
$\{3\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{0,2,3\}$

TABLE 6. Products of singletons in the complex algebra of $\mathfrak{K}_{28}$

Proof. Let $\mathfrak{K}_{28} = \langle K, R_{28}, {}^*, 0\rangle$, where $K = \{0,1,2,3\}$, $x^* = x$ for every $x \in K$, and R_{28} is the following ternary relation on K with 28 triples.

$$\begin{aligned} R_{28} :=& [0,0,0] \cup [1,1,1] \cup [2,2,2] \cup [3,3,3] \cup \\ & [0,1,1] \cup [0,2,2] \cup [0,3,3] \cup \\ & [1,2,2] \cup [3,1,1] \cup [2,3,3] \cup [1,2,3]. \end{aligned}$$

$\mathfrak{K}_{28}$ is (isomorphic to) the atom structure of the relation algebra $\mathsf{42}_{65}$ from (Maddux 2006). $\mathfrak{K}_{28}$ is a normal relevant model structure that satisfies (p1′) and the reflection properties (p5‴) and (p5⁗). By (p1′), the algebra of $\mathfrak{K}_{28}$ is the same as its complex algebra $\mathfrak{Cm}(\mathfrak{K}_{28})$. Neither (L″) nor (M″) is valid in $\mathfrak{K}_{28}$. Both (L″) and (M″) will fail if we choose variables $A, B, C, D, E, F, G \in \mathsf{Pv}$ and a valuation ν such that $\nu(A) = \{1\}$, $\nu(B) = \{1\}$, $\nu(C) = \{3\}$, $\nu(D) = \{2\}$, $\nu(E) = \{1\}$, $\nu(F) = \{3\}$, and $\nu(G) = \{1\}$. To check this it is convenient, in evaluating the terms in (L″) and (M″), to have the products of singletons in Table 6. By Theorem 12, we conclude that (L″) and (M″) are not provable in **R**. □

9. CONCLUSION

Algebras for relevance logic can be created in an abstract algebraic way: add operations for the connectives and distinguished elements for the constants, and impose on the operations and distinguished elements postulates that mimic the axioms. Operations in individual algebras may be specified by tables (in the finite case) or rules, and are designed to validate the axioms of the logic. Although algebraization may be mathematically illuminating, it is open to the philosophical charge that "... algebraic characterizations ... are merely formal, exhibiting no connection with the intended meanings of the logical constants" (Copeland 1979, p. 405).

Somewhat less abstract are the algebras of relevant model structures. Here the elements of the algebras are actually sets, so two of the operations, namely intersection and union, need not be specified by rules or postulates. But the other operations arise abstractly from the ternary relation R and the unary operation * of the structure. Postulates imposed on R and * are designed to validate the axioms. Indeed, many books and papers have lists of axioms (which are essentially second-order statements about relevant model structures) and their corresponding postulates on R and * (which are first-order statements about relevant model structures). Once again, "If the only constraint on * is that the resulting theory should validate the right set of sentences, then we are indeed in the presence of merely formal model theory" (Copeland 1979, p. 410).

In contrast, the elements of relational relevance algebras are binary relations, none of the operations are abstractly defined, and there are no postulates for R.

The operations of relational relevance algebras are just standard set-theoretically defined operations on binary relations. Of course, some axioms of **R** fail in R. The reasons for their failure are given in Theorem 4, from which we can see that the commutative dense relational relevance algebras will satisfy all the axioms of **R**. Focusing attention on the subclass of commutative dense algebras in R is a response to the axioms of **R**. For the system of Basic Logic consisting of axioms (A1)–(A20) and all nine rules, no such response is needed. The natural class of models is R, and Basic Logic is a finite approximation to R-logic.

One should expect *ad hoc* semantics ought to be sound and complete because they are designed for that purpose. But R-logic, R^{cd}-logic, $\mathsf{R}^{\mathsf{cdt}}$-logic, *etc.*, are part of the nineteenth century calculus of relations, while **R** and **RM** are mid-twentieth century inventions that just happen to be a proper subsystem of R^{cd}-logic and exactly the same as $\mathsf{R}^{\mathsf{cdt}}$-logic, respectively. Is this just a pure coincidence, or is there some underlying reason? There is no sign that the founders of relevance logic were trying to capture properties of binary relations in their axioms, so perhaps it is a coincidence. At least the binary relational interpretation escapes the charge that "... it is completely obscure what meaning is given to negation in the Routley-Meyer theory ..."(Copeland 1979, p. 408). The meaning of negation is quite clear; $\sim$ is converse-complementation. Anderson and Belnap (1975, p. 345) ask, "How then to interpret $\circ$? We confess puzzlement." In the binary relational interpretation, $\circ$ is composition.

Philosophical considerations are (or, at least, ought to be) constrained by mathematical theorems, so we give here a summary of the main results in this paper (Theorems 2, 3, 4, Corollary 5, and Theorems 7, 13, and 14).

$$(\mathrm{L}''), (\mathrm{M}'') \in \mathsf{R}\text{-logic} \subset \mathsf{R}^{\mathsf{cd}}\text{-logic} \subset \mathsf{R}^{\mathsf{cdt}}\text{-logic} = \mathbf{RM}$$
$$(\mathrm{L}''), (\mathrm{M}'') \notin \mathbf{R} \subset \mathsf{R}^{\mathsf{cd}}\text{-logic} \subset \{\mathbf{M}_0\}\text{-logic} = \mathbf{BM}$$
$$\mathsf{R}^{\mathsf{cd}}\text{-logic} \subset \{\mathbf{RM}84\}\text{-logic}$$

References

Ackermann, W. (1956). Begründung einer strengen Implikation. *J. Symb. Logic 21*, 113–128.

Anderson, A. R. and N. D. Belnap, Jr. (1975). *Entailment*. Princeton, N. J.: Princeton University Press.

Anderson, A. R., N. D. Belnap, Jr., and J. M. Dunn (1992). *Entailment. The logic of relevance and necessity. Vol. II.* Princeton, NJ: Princeton University Press.

Andréka, H., S. R. Givant, and I. Németi (1997). Decision problems for equational theories of relation algebras. *Mem. Amer. Math. Soc. 126*(604), xiv+126.

Belnap, Jr., N. D. (1960). Entailment and relevance. *J. Symbolic Logic 25*, 144–146.

Bimbo, K., J. M. Dunn, and R. D. Maddux (2008). Relevance logics and relation algebras. To appear in *The Review of Symbolic Logic.*

Brady, R. T. (Ed.) (2003). *Relevant Logics and their Rivals. Volume II.* Aldershot, Hants, England and Burlington, Vermont, USA: Ashgate Publishing Ltd.

Church, A. (1951). The weak positive implicational propositional calculus. *J. Symbolic Logic 16*(5), 238.

Copeland, B. J. (1979). On when a semantics is not a semantics: some reasons for disliking the Routley-Meyer semantics for relevance logic. *J. Philos. Logic 8*(4), 399–413.

De Morgan, A. (1856). On the symbols of logic, the theory of the syllogism, and in particular of the copula, and the application of the theory of probabilities to some questions in the theory of evidence. *Transactions of the Cambridge Philosophical Society 9*, 79–127. Reprinted in (De Morgan 1966).

De Morgan, A. (1864a). On the syllogism: III, and on logic in general. *Transactions of the Cambridge Philosophical Society 10*, 173–230. Reprinted in (De Morgan 1966).

De Morgan, A. (1864b). On the syllogism: IV, and on the logic of relations. *Transactions of the Cambridge Philosophical Society 10*, 331–358. Reprinted in (De Morgan 1966).

De Morgan, A. (1966). *"On the syllogism" and other logical writings*. New Haven, Conn.: Yale University Press.

Došen, K. (1992). The first axiomatization of relevant logic. *J. Philos. Logic 21*(4), 339–356.

Fine, K. (1974). Models for entailment. *J. Philos. Logic 3*, 347–372.

Kowalski, T. (2007). Weakly associative relation algebras hold the key to the universe. *Bulletin of the Section of Logic 36*(3/4), 145–157.

Lyndon, R. C. (1961). Relation algebras and projective geometries. *Michigan Math. J. 8*, 21–28.

Maddux, R. D. (1991). The origin of relation algebras in the development and axiomatization of the calculus of relations. *Studia Logica 50*(3-4), 421–455.

Maddux, R. D. (2006). *Relation Algebras*, Volume 150 of *Studies in Logic and Foundations of Mathematics*. Amsterdam: Elsevier B. V.

Maddux, R. D. (2007). Relevance logic and the calculus of relations (abstract). In *International Conference on Order, Algebra, and Logics, Vanderbilt University, June 13, 2007*, pp. 1–3.

Meyer, R. K. and R. Routley (1973). Classical relevant logics. I, II. *Studia Logica 32*, 51–68; ibid. 33 (1974), 183–194.

Mikulás, S. (2008). Algebras of relations and relevance logic. To appear in *J. Log. Comp.*

Moh, S.-K. (1950). The deduction theorems and two new logical systems. *Methodos 2*, 56–75.

Peirce, C. S. (1870). Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic. *Memoirs of the American Academy of Sciences 9*, 317–378. Reprinted in (Peirce 1960) and (Peirce 1984).

Peirce, C. S. (1880). On the algebra of logic. *American Journal of Mathematics 3*, 15–57. Reprinted in (Peirce 1960).

Peirce, C. S. (1883). Note B: the logic of relatives. In C. S. Peirce (Ed.), *Studies in Logic by Members of the Johns Hopkins University*, Boston, pp. 187–203. Little, Brown, and Co. Reprinted by John Benjamins Publishing Co., Amsterdam and Philadelphia, 1983, pp. lviii, vi+203.

Peirce, C. S. (1885). On the algebra of logic; a contribution to the philosophy of notation. *American Journal of Mathematics 7*, 180–202. Reprinted in (Peirce 1960).

Peirce, C. S. (1897). The logic of relatives. *The Monist 7*, 161–217. Reprinted in (Peirce 1960).

Peirce, C. S. (1960). *Collected Papers*. Cambridge, Mass.: The Belknap Press of Harvard University Press.

Peirce, C. S. (1984). *Writings of Charles S. Peirce. Vol. 2* (chronological ed.). Bloomington, IN: Indiana University Press.

Routley, R. and R. K. Meyer (1972a). The semantics of entailment. II. *J. Philos. Logic 1*(1), 53–73.

Routley, R. and R. K. Meyer (1972b). The semantics of entailment. III. *J. Philos. Logic 1*(2), 192–208.

Routley, R. and R. K. Meyer (1973). The semantics of entailment. I. In *Truth, Syntax and Modality (Proc. Conf. Alternative Semantics, Temple Univ., Philadelphia, Pa., 1970)*, pp. 199–243. Studies in Logic and the Foundations of Math., Vol. 68. Amsterdam: North-Holland.

Routley, R., V. Plumwood, R. K. Meyer, and R. T. Brady (1982). *Relevant Logics and their Rivals. Part I.* Atascadero, CA: Ridgeview Publishing Co.

Routley, R. and V. Routley (1972). The semantics of first degree entailment. *Noûs 6*(4), 335–359.

Schröder, F. W. K. E. (1966). *Vorlesungen über die Algebra der Logik (exacte Logik), Volume 3, "Algebra und Logik der Relative", Part I* (second ed.). Bronx, New York: Chelsea. First published by B. G. Teubner, Leipzig, 1895.

Sugihara, T. (1955). Strict implication free from implicational paradoxes. *Memoirs of the Faculty of Liberal Arts Series I*, 55–59.

Tarski, A. (1941). On the calculus of relations. *J. Symbolic Logic 6*, 73–89.

Tarski, A. and S. Givant (1987). *A formalization of set theory without variables.* Providence, RI: American Mathematical Society.

Urquhart, A. (1972). Semantics for relevant logics. *J. Symbolic Logic 37*, 159–169.

Urquhart, A. (1984). The undecidability of entailment and relevant implication. *J. Symbolic Logic 49*(4), 1059–1073.

Urquhart, A. (1996). Duality for algebras of relevant logics. *Studia Logica 56*(1-2), 263–276.

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