THE CLASSICAL RAMSEY NUMBER $R(3,3,3,3;2)$ IS NO GREATER THAN 62

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Abstract. In this paper we show that $R(3,3,3,3;2) \leq 62$, that is, any edge coloring of a complete graph on 62 vertices with four colors must contain a monochromatic triangle.

1. Introduction

An edge coloring of a complete graph is called *good* provided that there do not exist monochromatic triangles. There are, up to isomorphism, exactly two good edge colorings with three colors on the complete graph with 16 vertices. (See [5].) These are called the untwisted and twisted colorings. The automorphism group on each of these colorings acts transitively on the vertices. By removing a single vertex from each of these colorings (along with all edges incident with the removed vertex), we get two nonisomorphic good edge colorings with three colors on the complete graph with 15 vertices. We refer to these as the untwisted and twisted colorings on 15 vertices. There are no others, up to isomorphism. (See [4].)

All of the arguments in this paper are based on our intimate knowledge of the untwisted and twisted colorings on 15 and 16 vertices, together with the knowledge that these are the only such good edge colorings possible. If we knew the good edge colorings on 16 vertices, but not on 15 vertices, then the arguments in this paper (with trivial and obvious modifications) would suffice only to prove that $R(3,3,3,3;2) \leq 64$, instead of $R(3,3,3,3;2) \leq 62$, as we prove here. This would, however, still be an improvement on Folkman's Theorem (see [2]) that $R(3,3,3,3;2) \leq 65$.

In this paper, we improve the known upper bound for the classical Ramsey number $R(3,3,3,3;2)$. It is trivial to see that

$$\begin{aligned} (1)\quad R(3,3,3,3;2) &\le 4\cdot(R(3,3,3;2)-1)+1+1 \\ &= 4\cdot(17-1)+1+1 \\ &= 66. \end{aligned}$$

See [3]. In Folkman[2], it is shown that $R(3,3,3,3;2) \le 65$. In Sanchez-Flores[8], it is shown that $R(3,3,3,3;2) \le 64$. In this paper, we improve this to read $R(3,3,3,3;2) \le 62$. That is, we show that for any coloring of a complete graph with 62 vertices using four colors, there must exist a monochromatic triangle. The best known lower bound for $R(3,3,3,3;2)$ was provided by Chung[1], who constructed two nonisomorphic monochromatic triangle free edge colorings using four colors of the complete graph with 50 vertices, thus showing that $R(3,3,3,3;2) \ge 51$.

The paper is organized as follows. As noted below, the reader who wishes to get a feel for the global structure of the proof without reading the entire paper is advised to begin reading at chapter 4.

Chapter 1 contains an algebraic description, for readers unfamiliar with these colorings, of the untwisted and twisted good edge colorings on the complete graph on 16 vertices with three colors, in terms of the 16 element field $\mathbb{F}_{16}$. In the case of the twisted edge coloring, this description is new.

Chapter 2 contains the basic definitions needed to read this paper. Also included are some lemmas about the good edge colorings on the complete graph on 15 or 16 vertices with three colors, some without proof. Lemmas 9 thru 17 are needed only in sections 3.10, 3.11, and 3.12. The reader who wishes to read those sections, particularly the last two, should use colored pencils to draw the edge colorings in the conclusions of Lemmas 9 thru 13. This will involve drawing 55 edge colorings on the complete graph on 10 vertices and 5 edge colorings on the complete graph on 8 vertices, all with three colors. The reader who does this will find reading these sections quite easy. The reader who does not, will not. Some easy propositions about good edge colorings on the complete graph on 62 vertices with four colors are also proved here.

Chapter 3 is responsible for most of the length of this paper, and is best read with a good supply of colored pencils. It contains all of the local arguments. It's only purpose is to prove Theorem 13 at the beginning of chapter 4. The reader who believes Theorem 13 may skip this chapter. Suppose that we are given a good edge coloring, with four colors, on the complete graph on 62 vertices. Let u and v be two distinct vertices and let δ be any color. Suppose that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. (According to Definition 1 in chapter 2, we define the set $\|S_\delta(u)\|$ to be the set of all vertices x such that the edge from u to x is of color δ.) Then the set $S_\delta(u) \cap S_\delta(v)$ is refered to as an *attaching set*. The structure of a potential attaching set is quite limited. Our discussion of a potential attaching set is split into sections according to the cardinality of the attaching set. With only one exception, the sections of chapter 3 are independent of each other. The single exception is Theorem 11 in section 3.12, which uses Proposition 28 from section 3.10. Each section culminates in a single theorem, which, except for section 3.11, prohibits the existence of the attaching set in question. In section 3.11, the possibility of the existence of attaching sets of cardinality 5 is still allowed, but only under severely limited conditions. The longest and most complex sections of chapter 3 are section 3.11, which deals with attaching sets of cardinality 5, and section 3.12, which deals with attaching sets of cardinality 4. Except for Proposition 22, which makes use of Proposition 21, the propositions in section 3.11 are all independent of each other. Similarly, except for Proposition 28, which makes use of Proposition 27, the propositions in section 3.12 are all independent of each other.

Chapter 4 contains all of the global arguments, that is, arguments which consider more than one attaching set at a time. The reader who wants to get a feel for the global structure of the proof, without reading the entire paper, is strongly advised to begin reading here, accepting Theorem 13 on faith and refering to the definitions and propositions of chapter 2 only as needed. (Not much is needed here.)

Theorem 13 states that all attaching sets in a good edge coloring on the complete graph on 62 vertices with four colors must have cardinalities 0, 1, 2, or 5. Furthermore, attaching sets of cardinality 5 are severely restricted. As stated above, the sole purpose of chapter 3 is to prove this theorem. From this theorem alone, with the help of rather trivial results from chapter 2, we prove our main result, Theorem 17, that $R(3,3,3,3;2) \leq 62$.

The writing up of these results was greatly facilitated by a streamlining of the global arguments. The original proof required consideration of generalized attaching sets where one or (shudder!) both of the sets $S_\delta(u)$ and $S_\delta(v)$ were allowed to have cardinality 15. The additional local arguments in that proof were significantly more involved than the ones in chapter 3, and their inclusion would have resulted in that section's being several times its current lenght. As the task of converting several hundred pages of four colored drawings with colored pencils to the monochromatic proof presented here was already quite formidable, the author is indeed pleased that this streamlining occured.

2. Good Colorings with Three Colors: An Algebraic Description of the Untwisted and Twisted Colorings

In this section, we give algebraic descriptions, without proofs, of both the untwisted and the twisted colorings on 16 vertices. In both models, we use the elements of $\mathbb{F}_{16}$ as the vertices and describe the colorings in terms of symmetric polynomials in $\mathbb{F}_2[x,y]$ which define maps

$$\mathbb{F}_{16} \times \mathbb{F}_{16} \longrightarrow \mathbb{F}_4.$$

The specific polynomial maps that we choose are not the only ones possible.

Recall that $\mathbb{F}_{16}$ is the splitting field of the polynomial

$$x^{16} + x = x(x+1)(x^2+x+1)(x^4+x^3+x^2+x+1)(x^4+x^3+1)(x^4+x+1)$$

over $\mathbb{F}_2$. The subfield $\mathbb{F}_4$ is the splitting field of the polynomial

$$x^4 + x = x(x+1)(x_2+x+1)$$

over $\mathbb{F}_2$. The three colors will be represented by the nonzero elements of $\mathbb{F}_4$, that is, the zeros of $x^3+1 = (x+1)(x^2+x+1)$. If $a^2+a+1=0$ for some $a \in \mathbb{F}_{16}$, then the set of colors is $\{a, a^2, a^3 = 1\}$.

For any $i \in \{0,1\}$ we define the symmetric polynomial $\Gamma_i(x,y) \in \mathbb{F}_2[x,y]$ by

$$\Gamma_0(x,y) = (x+y)^{10}$$

and

$$\Gamma_1(x,y) = (x+y)^{10} + (x+x^4+y+y^4)(x+x^2+x^4+x^8)(y+y^2+y^4+y^8).$$

It is easy to see, for any $i \in \{0,1\}$, that $\Gamma_i(x,y)$ defines a map from $\mathbb{F}_{16} \times \mathbb{F}_{16} \longrightarrow \mathbb{F}_4$ such that $\Gamma_i(a,b) = 0$ iff $a = b$ for any $a, b \in \mathbb{F}_{16}$. Thus, $\Gamma_i(x,y)$ defines an edge coloring for any $i \in \{0,1\}$, with $\mathbb{F}_4 \sim \{0\}$ as the set of colors.

$\Gamma_0(x,y)$ describes the untwisted coloring. The (possibly color permuting) automorphisms are generated by the following maps $\Theta : \mathbb{F}_{16} \longrightarrow \mathbb{F}_{16}$.

(1) The map $\Theta(x) = x + a$ for any $a \in \mathbb{F}_{16}$ satisfies

$$\Gamma_0(\Theta(x), \Theta(y)) = \Gamma_0(x,y)$$

in $\mathbb{F}_{16}$.

(2) The map $\Theta(x) = ax$ for any $a \in \mathbb{F}_{16}$ satisfies

$$\Gamma_0(\Theta(x), \Theta(y)) = a^{10}\Gamma_0(x,y)$$

in $\mathbb{F}_{16}$.

(3) The map $\Theta(x) = x^2$ satisfies

$$\Gamma_0(\Theta(x), \Theta(y)) = \Gamma_0(x, y)^2$$

in $\mathbb{F}_{16}$.

This is the model given in [3] for the untwisted coloring, although its description as a polynomial is not explicitly given there.

$\Gamma_1(x, y)$ describes the twisted coloring. The (possibly color permuting) automorphisms are generated by the following maps $\Theta : \mathbb{F}_{16} \longrightarrow \mathbb{F}_{16}$.

(1) The map $\Theta(x) = x + a$ for any $a \in \mathbb{F}_{16}$ such that $a(a+1)(a^2+a+2)(a^4+a+1) = 0$ satisfies

$$\Gamma_1(\Theta(x), \Theta(y)) = \Gamma_1(x, y)$$

in $\mathbb{F}_{16}$.

(2) The map $\Theta(x) = x^2$ satisfies

$$\Gamma_1(\Theta(x), \Theta(y)) = \Gamma_1(x, y)^2$$

in $\mathbb{F}_{16}$.

(3) The map $\Theta(x) = a^2 + ax + a^2(x^4 + x + a)^3$ for any $a \in \mathbb{F}_{16}$ such that $a^2 + a + 1 = 0$ satisfies

$$\Gamma_1(\Theta(x), \Theta(y)) = a\Gamma_1(x, y)$$

in $\mathbb{F}_{16}$.

This model for the twisted coloring is new, and is motivated by anology with the above model for the untwisted coloring and by considerations from [6].

3. Preliminaries

For convenience, we define

$$[i_1, \ldots, i_n] = \{\, (i_{f(1)}, \ldots, i_{f(n)}) \mid f \text{ is a permutation on } \{1, \ldots, n\} \,\}.$$

We also write $u \stackrel{\alpha}{\longrightarrow} v$, where u and v are vertices in some edge colored graph and α is a color, to indicate that the edge connecting u and v is of color α.

Definition 1. *Let V be the vertex set of an edge colored complete graph. Let α be a color and let $v \in V$. Then we define*

$$S_\alpha(v) = \{\, x \in V \mid x \stackrel{\alpha}{\longrightarrow} v \,\}.$$

Definition 2. *Let $v_0, \ldots, v_{15}$ be vertices of an edge coloring of a complete graph, and let α, β, and γ be colors. We define the following predicates:*

(1) $P_{\beta,\alpha}(v_1, \ldots, v_5)$ *iff*$_{df}$ $v_1 \stackrel{\beta}{\longrightarrow} v_2 \stackrel{\beta}{\longrightarrow} v_3 \stackrel{\beta}{\longrightarrow} v_4 \stackrel{\beta}{\longrightarrow} v_5 \stackrel{\beta}{\longrightarrow} v_1$
and $v_1 \stackrel{\alpha}{\longrightarrow} v_3 \stackrel{\alpha}{\longrightarrow} v_5 \stackrel{\alpha}{\longrightarrow} v_2 \stackrel{\alpha}{\longrightarrow} v_4 \stackrel{\alpha}{\longrightarrow} v_1$.

(2) $M^0_{\beta,\alpha,\gamma}(v_1, \ldots, v_{10})$ *iff*$_{df}$ $P_{\beta,\alpha}(v_1, \ldots, v_5)$ *and* $P_{\beta,\alpha}(v_6, \ldots, v_{10})$
and $v_3 \stackrel{\gamma}{\longrightarrow} v_6, v_7, v_9, v_{10}$ *and* $v_8 \stackrel{\gamma}{\longrightarrow} v_1, v_2, v_4, v_5$ *and* $v_1 \stackrel{\gamma}{\longrightarrow} v_6$
and $v_2 \stackrel{\gamma}{\longrightarrow} v_7$ *and* $v_3 \stackrel{\gamma}{\longrightarrow} v_8$ *and* $v_4 \stackrel{\gamma}{\longrightarrow} v_9$ *and* $v_5 \stackrel{\gamma}{\longrightarrow} v_{10}$
and $v_1 \stackrel{\beta}{\longrightarrow} v_7 \stackrel{\beta}{\longrightarrow} v_4 \stackrel{\beta}{\longrightarrow} v_{10}$ *and* $v_6 \stackrel{\beta}{\longrightarrow} v_2 \stackrel{\beta}{\longrightarrow} v_9 \stackrel{\beta}{\longrightarrow} v_5$
and $v_2 \stackrel{\alpha}{\longrightarrow} v_{10} \stackrel{\alpha}{\longrightarrow} v_1 \stackrel{\alpha}{\longrightarrow} v_9$ *and* $v_7 \stackrel{\alpha}{\longrightarrow} v_5 \stackrel{\alpha}{\longrightarrow} v_6 \stackrel{\alpha}{\longrightarrow} v_4$.

(3) $M^1_{\beta,\alpha,\gamma}(v_1, \ldots, v_{10})$ *iff*$_{df}$ $P_{\beta,\alpha}(v_1, \ldots, v_5)$ *and* $P_{\beta,\alpha}(v_6, \ldots, v_{10})$
and $v_4 \stackrel{\gamma}{\longrightarrow} v_6, v_8, v_9, v_{10}$ *and* $v_7 \stackrel{\gamma}{\longrightarrow} v_1, v_2, v_3, v_5$ *and* $v_2 \stackrel{\gamma}{\longrightarrow} v_9$
and $v_4 \stackrel{\gamma}{\longrightarrow} v_7$ *and* $v_1 \stackrel{\gamma}{\longrightarrow} v_6$ *and* $v_3 \stackrel{\gamma}{\longrightarrow} v_8$ *and* $v_5 \stackrel{\gamma}{\longrightarrow} v_{10}$
and $v_1 \stackrel{\beta}{\longrightarrow} v_{10}$ *and* $v_2 \stackrel{\beta}{\longrightarrow} v_8 \stackrel{\beta}{\longrightarrow} v_5 \stackrel{\beta}{\longrightarrow} v_6 \stackrel{\beta}{\longrightarrow} v_3 \stackrel{\beta}{\longrightarrow} v_9$
and $v_3 \stackrel{\alpha}{\longrightarrow} v_{10} \stackrel{\alpha}{\longrightarrow} v_2 \stackrel{\alpha}{\longrightarrow} v_6$ *and* $v_5 \stackrel{\alpha}{\longrightarrow} v_9 \stackrel{\alpha}{\longrightarrow} v_1 \stackrel{\alpha}{\longrightarrow} v_8$.

(4) $M^2_{\beta,\alpha,\gamma}(v_1,\dots,v_{10})$ *iff*$_{df}$ $P_{\beta,\alpha}(v_1,\dots,v_5)$ *and* $P_{\beta,\alpha}(v_6,\dots,v_{10})$
and $v_5 \overset{\gamma}{\rightarrow}\text{——} v_7, v_8, v_9, v_{10}$ *and* $v_6 \overset{\gamma}{\rightarrow}\text{——} v_1, v_2, v_3, v_4$ *and* $v_1 \overset{\gamma}{\rightarrow}\text{——} v_{10}$
and $v_5 \overset{\gamma}{\rightarrow}\text{——} v_6$ *and* $v_2 \overset{\gamma}{\rightarrow}\text{——} v_7$ *and* $v_3 \overset{\gamma}{\rightarrow}\text{——} v_8$ *and* $v_4 \overset{\gamma}{\rightarrow}\text{——} v_9$
and $v_2 \overset{\beta}{\rightarrow}\text{——} v_{10} \overset{\beta}{\rightarrow}\text{——} v_4 \overset{\beta}{\rightarrow}\text{——} v_8$ *and* $v_3 \overset{\beta}{\rightarrow}\text{——} v_7 \overset{\beta}{\rightarrow}\text{——} v_1 \overset{\beta}{\rightarrow}\text{——} v_9$
and $v_4 \overset{\alpha}{\rightarrow}\text{——} v_7$ *and* $v_1 \overset{\alpha}{\rightarrow}\text{——} v_8 \overset{\alpha}{\rightarrow}\text{——} v_2 \overset{\alpha}{\rightarrow}\text{——} v_9 \overset{\alpha}{\rightarrow}\text{——} v_3 \overset{\alpha}{\rightarrow}\text{——} v_{10}$.

(5) $N^0_{\beta,\alpha,\gamma}(v_1,\dots,v_8)$ *iff*$_{df}$ $P_{\gamma,\alpha}(v_1, v_2, v_3, v_4, v_5)$
and $P_{\gamma,\alpha}(v_5, v_6, v_7, v_8, v_1)$ *and* $v_7 \overset{\gamma}{\rightarrow}\text{——} v_3$ *and* $v_8 \overset{\gamma}{\rightarrow}\text{——} v_4$
and $v_6 \overset{\gamma}{\rightarrow}\text{——} v_2$ *and* $v_8 \overset{\beta}{\rightarrow}\text{——} v_2 \overset{\beta}{\rightarrow}\text{——} v_7 \overset{\beta}{\rightarrow}\text{——} v_4 \overset{\beta}{\rightarrow}\text{——} v_6 \overset{\beta}{\rightarrow}\text{——} v_3 \overset{\beta}{\rightarrow}\text{——} v_8$.

(6) $N^1_{\beta,\alpha,\gamma}(v_1,\dots,v_8)$ *iff*$_{df}$ $P_{\gamma,\alpha}(v_1, v_2, v_3, v_4, v_5)$
and $P_{\gamma,\alpha}(v_5, v_6, v_7, v_8, v_1)$ *and* $v_7 \overset{\gamma}{\rightarrow}\text{——} v_3$ *and* $v_8 \overset{\gamma}{\rightarrow}\text{——} v_4$
and $v_8 \overset{\beta}{\rightarrow}\text{——} v_2 \overset{\beta}{\rightarrow}\text{——} v_7 \overset{\beta}{\rightarrow}\text{——} v_4 \overset{\beta}{\rightarrow}\text{——} v_6 \overset{\beta}{\rightarrow}\text{——} v_3 \overset{\beta}{\rightarrow}\text{——} v_8$ *and* $v_6 \overset{\alpha}{\rightarrow}\text{——} v_2$.

(7) $N^2_{\beta,\alpha,\gamma}(v_1,\dots,v_8)$ *iff*$_{df}$ $P_{\gamma,\alpha}(v_1, v_2, v_3, v_4, v_5)$
and $P_{\gamma,\alpha}(v_5, v_6, v_7, v_8, v_1)$ *and* $v_8 \overset{\gamma}{\rightarrow}\text{——} v_4$ *and* $v_7 \overset{\gamma}{\rightarrow}\text{——} v_2$
and $v_6 \overset{\gamma}{\rightarrow}\text{——} v_3$ *and* $v_8 \overset{\beta}{\rightarrow}\text{——} v_2 \overset{\beta}{\rightarrow}\text{——} v_6 \overset{\beta}{\rightarrow}\text{——} v_4 \overset{\beta}{\rightarrow}\text{——} v_7 \overset{\beta}{\rightarrow}\text{——} v_3 \overset{\beta}{\rightarrow}\text{——} v_8$.

(8) $A^0_{\alpha,\beta,\gamma}(v_1,\dots,v_{10})$ *iff* $_{df}$ $P_{\beta,\alpha}(v_1, v_2, v_3, v_4, v_5)$
and $P_{\gamma,\beta}(v_6, v_7, v_8, v_9, v_{10})$ *and* $v_1 \overset{\beta}{\rightarrow}\text{——} v_6$ *and* $v_2 \overset{\beta}{\rightarrow}\text{——} v_7$
and $v_3 \overset{\beta}{\rightarrow}\text{——} v_8$ *and* $v_4 \overset{\beta}{\rightarrow}\text{——} v_9$ *and* $v_5 \overset{\beta}{\rightarrow}\text{——} v_{10}$ *and* $v_1 \overset{\alpha}{\rightarrow}\text{——} v_8 \overset{\alpha}{\rightarrow}\text{——} v_5 \overset{\alpha}{\rightarrow}\text{——} v_7 \overset{\alpha}{\rightarrow}\text{——} v_4 \overset{\alpha}{\rightarrow}\text{——} v_6 \overset{\alpha}{\rightarrow}\text{——} v_3 \overset{\alpha}{\rightarrow}\text{——} v_{10} \overset{\alpha}{\rightarrow}\text{——} v_2 \overset{\alpha}{\rightarrow}\text{——} v_9 \overset{\alpha}{\rightarrow}\text{——} v_1$
and $v_1 \overset{\gamma}{\rightarrow}\text{——} v_7 \overset{\gamma}{\rightarrow}\text{——} v_3 \overset{\gamma}{\rightarrow}\text{——} v_9 \overset{\gamma}{\rightarrow}\text{——} v_5 \overset{\gamma}{\rightarrow}\text{——} v_6 \overset{\gamma}{\rightarrow}\text{——} v_2 \overset{\gamma}{\rightarrow}\text{——} v_8 \overset{\gamma}{\rightarrow}\text{——} v_4$
$\overset{\gamma}{\rightarrow}\text{——} v_{10} \overset{\gamma}{\rightarrow}\text{——} v_1$.

(9) $A^1_{\alpha,\beta,\gamma}(v_1,\dots,v_{10})$ *iff*$_{df}$ $P_{\beta,\alpha}(v_1, v_2, v_3, v_4, v_5)$
and $P_{\gamma,\beta}(v_6, v_7, v_8, v_9, v_{10})$ *and* $v_1 \overset{\beta}{\rightarrow}\text{——} v_6$ *and* $v_2 \overset{\beta}{\rightarrow}\text{——} v_7$
and $v_3 \overset{\beta}{\rightarrow}\text{——} v_8$ *and* $v_4 \overset{\beta}{\rightarrow}\text{——} v_9$ *and* $v_5 \overset{\beta}{\rightarrow}\text{——} v_{10}$ *and* $v_1 \overset{\alpha}{\rightarrow}\text{——} v_8 \overset{\alpha}{\rightarrow}\text{——} v_5 \overset{\alpha}{\rightarrow}\text{——} v_7 \overset{\alpha}{\rightarrow}\text{——} v_4 \overset{\alpha}{\rightarrow}\text{——} v_{10} \overset{\alpha}{\rightarrow}\text{——} v_3 \overset{\alpha}{\rightarrow}\text{——} v_6 \overset{\alpha}{\rightarrow}\text{——} v_2 \overset{\alpha}{\rightarrow}\text{——} v_9 \overset{\alpha}{\rightarrow}\text{——} v_1$
and $v_1 \overset{\gamma}{\rightarrow}\text{——} v_7 \overset{\gamma}{\rightarrow}\text{——} v_3 \overset{\gamma}{\rightarrow}\text{——} v_9 \overset{\gamma}{\rightarrow}\text{——} v_5 \overset{\gamma}{\rightarrow}\text{——} v_6 \overset{\gamma}{\rightarrow}\text{——} v_4 \overset{\gamma}{\rightarrow}\text{——} v_8 \overset{\gamma}{\rightarrow}\text{——} v_2$
$\overset{\gamma}{\rightarrow}\text{——} v_{10} \overset{\gamma}{\rightarrow}\text{——} v_1$.

(10) $C^0_{\alpha,\beta,\gamma}(v_1,\dots,v_{15})$ *iff*$_{df}$ $A^0_{\alpha,\beta,\gamma}(v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1, v_2, v_3, v_4, v_5)$
and $A^0_{\beta,\gamma,\alpha}(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$
and $A^0_{\gamma,\alpha,\beta}(v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15})$.

(11) $C^1_{\alpha,\beta,\gamma}(v_1,\dots,v_{15})$ *iff*$_{df}$ $A^1_{\alpha,\beta,\gamma}(v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1, v_2, v_3, v_4, v_5)$
and $A^1_{\beta,\gamma,\alpha}(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$
and $A^1_{\gamma,\alpha,\beta}(v_6, v_7, v_8, v_9, v_{10}, v_{15}, v_{14}, v_{13}, v_{12}, v_{11})$.

(12) $B^0_{\alpha,\beta,\gamma}(v_0,\dots,v_{15})$ *iff*$_{df}$ $C^0_{\alpha,\beta,\gamma}(v_1,\dots,v_{15})$
and $v_0 \overset{\alpha}{\rightarrow}\text{——} v_1, v_2, v_3, v_4, v_5$ *and* $v_0 \overset{\beta}{\rightarrow}\text{——} v_6, v_7, v_8, v_9, v_{10}$
and $v_0 \overset{\gamma}{\rightarrow}\text{——} v_{11}, v_{12}, v_{13}, v_{14}, v_{15}$.

$$(13)\quad B^1_{\alpha,\beta,\gamma}(v_0,\ldots,v_{15}) \qquad \textit{iff}_{df} \qquad C^1_{\alpha,\beta,\gamma}(v_1,\ldots,v_{15})$$
$$\textit{and} \quad v_0 \overset{\alpha}{\to} \text{——} v_1,v_2,v_3,v_4,v_5 \quad \textit{and} \quad v_0 \overset{\beta}{\to} \text{——} v_6,v_7,v_8,v_9,v_{10}$$
$$\textit{and} \quad v_0 \overset{\gamma}{\to} \text{——} v_{11},v_{12},v_{13},v_{14},v_{15}.$$

Lemma 1. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, and γ. If $\|U\| = 16$, then there exist $x_0,\ldots,x_{15} \in U$ and some $i \in \{0,1\}$ such that $B^i_{\alpha,\beta,\gamma}(x_0,\ldots,x_{15})$*

Proof. See J. G. Kalbfleisch and R. G. Stanton [5]. □

Lemma 2. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, and γ. If $\|U\| = 15$, then there exist $x_1,\ldots,x_{15} \in U$ and some $i \in \{0,1\}$ such that $C^i_{\alpha,\beta,\gamma}(x_1,\ldots,x_{15})$*

Remark In Lemma 2, $i = 0$ if the coloring is untwisted, and $i = 1$ if the coloring is twisted.

Proof. See K. Heinrich [4]. □

Lemma 3. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, and δ. Let $A, B \subseteq U$ both contain no edges of color δ. Suppose that $\|U\| \geq 15$ and $\|A\| = \|B\| = 5$. Then $\|A \cap B\| \in \{0, 2, 5\}$.*

Proof. Clearly, $\|U\| \leq R(3,3,3;2) - 1 = 17 - 1 = 16$ so that the conclusion follows from Lemma 1 and Lemma 2 by inspection. □

Proposition 1. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $v \in V$. Then*

$$(2) \quad \big(\|S_\alpha(v)\|, \|S_\beta(v)\|, \|S_\gamma(v)\|, \|S_\delta(v)\|\big) \in [16,16,16,13] \cup [16,16,15,14] \cup [16,15,15,15].$$

Proof. It is clear that

$$\begin{aligned} 62 &= \|V\| \\ &= \|\{v\} \uplus S_\alpha(v) \uplus S_\beta(v) \uplus S_\gamma(v) \uplus S_\delta(v)\| \\ &= 1 + \|S_\alpha(v)\| + \|S_\beta(v)\| + \|S_\gamma(v)\| + \|S_\delta(v)\|. \end{aligned}$$

Also, for any $\eta \in \{\alpha, \beta, \gamma, \delta\}$, we see that the induced good edge coloring on the complete graph with vertex set $S_\eta(v)$ cannot contain any edges of color η, since otherwise we would have a monochromatic triangle of color η in V, contradicting the goodness of the original coloring. Thus, we have

$$\|S_\alpha(v)\|, \|S_\beta(v)\|, \|S_\gamma(v)\|, \|S_\delta(v)\| \leq R(3,3,3;2) - 1 = 17 - 1 = 16.$$

The proposition follows. □

Proposition 2. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $v \in V$ with $\|S_\delta(v)\| = 16$. Then for any $x \in S_\delta(v)$ we have*

$$\|S_\delta(v) \cap S_\alpha(x)\| = \|S_\delta(v) \cap S_\beta(x)\| = \|S_\delta(v) \cap S_\gamma(x)\| = 5.$$

Proof. Note that $S_\delta(v) \cap S_\delta(x) = \emptyset$, since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned}16 &= \|S_\delta(v)\| \\ &= \big\|\{x\} \uplus \big(S_\delta(v) \cap S_\alpha(x)\big) \uplus \big(S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(v) \cap S_\gamma(x)\big)\big\| \\ &= 1 + \big\|S_\delta(v) \cap S_\alpha(x)\big\| + \big\|S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(v) \cap S_\gamma(x)\big\|.\end{aligned}$$

But, since $S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ or η whenever $\eta \in \{\alpha, \beta, \gamma\}$, we see that

$$\|S_\delta(v) \cap S_\eta(x)\| \le R(3,3;2) - 1 = 6 - 1 = 5$$

for any $\eta \in \{\alpha, \beta, \gamma\}$. The proposition follows. □

Proposition 3. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. If $\eta \in \{\alpha, \beta, \gamma\}$ and $x \in S_\delta(u) \cap S_\delta(v)$ with $\|S_\eta(x)\| \ge 15$, then*

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{0, 2, 5\}.$$

Proof. Without loss of generality, we assume that $\eta = \gamma$. Let $U = S_\gamma(x)$. Let $A = S_\delta(u) \cap S_\gamma(x)$ and $B = S_\delta(v) \cap S_\gamma(x)$. Note that $A, B \subseteq U$. Also, the complete graph with vertex set U has no edges of color γ under the induced good edge coloring, and thus is colored with the colors α, β, and δ. Also, neither A nor B has any edges of color δ. Furthermore, $\|U\| \ge 15$. By Proposition 2, we see that $\|A\| = \|B\| = 5$. We may now apply Lemma 3 to see that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = \|A \cap B\| \in \{0, 2, 5\},$$

as desired. □

Proposition 4. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big\|\{\, \eta \in \{\alpha, \beta, \gamma\} \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{0, 2, 5\} \,\}\big\| \ge 2.$$

Proof. First note that

$$\begin{aligned}(3) \quad &\{\, \eta \in \{\alpha, \beta, \gamma\} \mid \|S_\eta(x)\| \ge 15 \,\} \\ &\qquad\qquad \subseteq \{\, \eta \in \{\alpha, \beta, \gamma\} \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{0, 2, 5\} \,\}\end{aligned}$$

by Proposition 3. But

$$\big\|\{\, \eta \in \{\alpha, \beta, \gamma\} \mid \|S_\eta(x)\| \ge 15 \,\}\big\| \ge 2$$

by Proposition 1. The proposition follows. □

Lemma 4. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, and γ. Let $\|U\| = 16$ and let $a \in U$. Then $\|\{\, x \in U \mid x \overset{\alpha}{\to} \!\!-\!\!-\, a \,\}\| = 5$. Furthermore, there exists $w_0, w_1, w_2, w_3, w_4 \in U$ such that $\{w_0, w_1, w_2, w_3, w_4\} = \{\, x \in U \mid x \overset{\alpha}{\to} \!\!-\!\!-\, a \,\}$ and $P_{\beta,\gamma}(w_0, w_1, w_2, w_3, w_4)$. Moreover, if $x, y \in U$ satisfy $a \overset{\alpha}{\to} \!\!-\!\!-\, x \overset{\beta}{\to} \!\!-\!\!-\, y \overset{\alpha}{\to} \!\!-\!\!-\, a$, we may assume that $x = w_0$ and $y = w_4$.*

Proof. We omit the proof. □

Lemma 5. *Let U be the vertex set of a complete graph with a good edge coloring with the pairwise distinct colors α, β, and γ, with $\|U\| = 16$. Let $a, b \in U$ with $a \overset{\alpha}{\to} \!\!-\!\!-\, b$. Then there exist $x_0, \ldots, x_{15} \in U$ such that one of the following holds:*

(1) $C^0_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ *with $a = x_0$ and $b = x_3$.*

(2) $C^1_{\alpha,\beta,\gamma}(x_0,\dots,x_{15})$ *with* $a = x_0$ *and* $b = x_3$.
(3) $C^1_{\alpha,\beta,\gamma}(x_0,\dots,x_{15})$ *with* $a = x_0$ *and* $b = x_2$.
(4) $C^1_{\alpha,\beta,\gamma}(x_0,\dots,x_{15})$ *with* $a = x_0$ *and* $b = x_1$.

Remark In Lemma 5, (1) applies if the coloring is untwisted, and either (2) or (3) or (4) applies if the coloring is twisted.

Proof. We omit the proof. ☐

Lemma 6. *Let* U *be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors* α, β, *and* γ. *Let* $\|U\| = 16$ *and let* $a, b \in U$ *with* $a \stackrel{\gamma}{\to} \text{——}\, b$. *Then, we have the following:*

(1) $\|\{\, x \in U \mid a \stackrel{\alpha}{\to} \text{——}\, x \stackrel{\alpha}{\to} \text{——}\, b \,\}\| = 2$
(2) $\|\{\, x \in U \mid a \stackrel{\alpha}{\to} \text{——}\, x \stackrel{\beta}{\to} \text{——}\, b \,\}\| = 1$
(3) $\|\{\, x \in U \mid a \stackrel{\alpha}{\to} \text{——}\, x \stackrel{\gamma}{\to} \text{——}\, b \,\}\| = 2$

Proof. We omit the proof. ☐

Lemma 7. *Let* U *be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors* α, β, *and* γ. *Suppose that* $\|U\| = 16$ *and* $\eta \in \{\alpha, \beta, \gamma\}$. *If* $a, b, c, d, e \in U$ *are pairwise distinct and satisfy* $a \stackrel{\eta}{\to} \text{——}\, b \stackrel{\eta}{\to} \text{——}\, c$ *and* $d \stackrel{\eta}{\to} \text{——}\, e$, *then there exists some* $x \in \{a, b, c\}$ *and some* $y \in \{d, e\}$ *such that* $x \stackrel{\eta}{\to} \text{——}\, y$.

Proof. Suppose not. Then there exist distinct $x_0, x_1 \in U$ with $b \stackrel{\eta}{\to} \text{——}\, x_0, x_1 \stackrel{\eta}{\to} \text{——}\, d$ and there exist distinct $y_0, y_1 \in U$ with $b \stackrel{\eta}{\to} \text{——}\, y_0, y_1 \stackrel{\eta}{\to} \text{——}\, e$. But $\{x_0, x_1\} \cap \{y_0, y_1\} = \emptyset$, since otherwise we would have a monochromatic triangle. Clearly $a, c \notin \{x_0, x_1, y_0, y_1\}$, since we cannot have any η colored edges from $\{a, c\}$ to $\{d, e\}$. thus, we have pairwise distinct elements $a, c, x_0, x_1, y_0, y_1 \in U$ with $b \stackrel{\eta}{\to} \text{——}\, a, c, x_0, x_1, y_0, y_1$, which is impossible, by Lemma 3. ☐

Lemma 8. *Let* U *be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors* α, β, *and* γ. *Suppose that* $\|U\| = 16$ *and* $\eta \in \{\alpha, \beta, \gamma\}$. *If* $a, b, c, d \in U$ *are four pairwise distinct elements which satisfy* $a \stackrel{\eta}{\to} \text{——}\, b \stackrel{\eta}{\to} \text{——}\, c \stackrel{\eta}{\to} \text{——}\, d \stackrel{\eta}{\to} \text{——}\, a$, *then for any* $x \in U \sim \{a, b, c, d\}$ *there exists a unique* $y \in \{a, b, c, d\}$ *such that* $x \stackrel{\eta}{\to} \text{——}\, y$.

Proof. By lemma 4, there exist pairwise distinct $x_0, x_1, x_2 \in U \sim \{d, b\}$ such that $a \stackrel{\eta}{\to} \text{——}\, b, d, x_0, x_1, x_2$. Similarly, there exist pairwise distinct $y_0, y_1, y_2 \in U \sim \{a, c\}$ such that $b \stackrel{\eta}{\to} \text{——}\, a, c, y_0, y_1, y_2$, pairwise distinct $z_0, z_1, z_2 \in U \sim \{b, d\}$ such that $c \stackrel{\eta}{\to} \text{——}\, b, d, z_0, z_1, z_2$, and $w_0, w_1, w_2 \in U \sim \{c, a\}$ such that $d \stackrel{\eta}{\to} \text{——}\, c, a, w_0, w_1, w_2$.

Note that the elements $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, w_0, w_1, w_2$ must be pairwise distinct, since otherwise we would either get a monochromatic triangle, or else a violation of Lemma 6(1). Thus, we have

$$U = \{a, b, c, d, x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, w_0, w_1, w_2\}.$$

The lemma follows. ☐

Lemma 9. *Let* $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ *be vertices in some good edge coloring of a complete graph on either* 15 *or* 16 *vertices, colored with the three pairwise distinct colors* α, β, γ. *Suppose that* *the following hold:*

$$\{u_0,u_1,u_2,u_3,u_4\}\cap\{v_0,v_1,v_2,v_3,v_4\}=\emptyset$$
$$P_{\beta,\alpha}(u_3,u_4,u_0,u_1,u_2)$$
$$P_{\beta,\alpha}(v_3,v_4,v_0,v_1,v_2)$$
$$u_3 \stackrel{\gamma}{\rightarrow} \text{——}\, v_3$$
$$u_4 \stackrel{\gamma}{\rightarrow} \text{——}\, v_4$$
$$u_1 \stackrel{\gamma}{\rightarrow} \text{——}\, v_1$$
$$u_2 \stackrel{\gamma}{\rightarrow} \text{——}\, v_2$$

Then $u_0 \stackrel{\gamma}{\rightarrow} \text{——}\, v_0$*. Furthermore, one of the following 25 cases must hold:*

(1) $M^0_{\beta,\alpha,\gamma}(u_0,u_1,u_2,u_3,u_4,v_0,v_1,v_2,v_3,v_4)$
(2) $M^0_{\beta,\alpha,\gamma}(u_1,u_2,u_3,u_4,u_0,v_1,v_2,v_3,v_4,v_0)$
(3) $M^0_{\beta,\alpha,\gamma}(u_2,u_3,u_4,u_0,u_1,v_2,v_3,v_4,v_0,v_2)$
(4) $M^0_{\beta,\alpha,\gamma}(u_3,u_4,u_0,u_1,u_2,v_3,v_4,v_0,v_1,v_2)$
(5) $M^0_{\beta,\alpha,\gamma}(u_4,u_0,u_1,u_2,u_3,v_4,v_0,v_1,v_2,v_3)$
(6) $M^1_{\beta,\alpha,\gamma}(u_0,u_1,u_2,u_3,u_4,v_0,v_1,v_2,v_3,v_4)$
(7) $M^1_{\beta,\alpha,\gamma}(u_1,u_2,u_3,u_4,u_0,v_1,v_2,v_3,v_4,v_0)$
(8) $M^1_{\beta,\alpha,\gamma}(u_2,u_3,u_4,u_0,u_1,v_2,v_3,v_4,v_0,v_1)$
(9) $M^1_{\beta,\alpha,\gamma}(u_3,u_4,u_0,u_1,u_2,v_3,v_4,v_0,v_1,v_2)$
(10) $M^1_{\beta,\alpha,\gamma}(u_4,u_0,u_1,u_2,u_3,v_4,v_0,v_1,v_2,v_3)$
(11) $M^1_{\beta,\alpha,\gamma}(u_4,u_3,u_2,u_1,u_0,v_4,v_3,v_2,v_1,v_0)$
(12) $M^1_{\beta,\alpha,\gamma}(u_3,u_2,u_1,u_0,u_4,v_3,v_2,v_1,v_0,v_4)$
(13) $M^1_{\beta,\alpha,\gamma}(u_2,u_1,u_0,u_4,u_3,v_2,v_1,v_0,v_4,v_3)$
(14) $M^1_{\beta,\alpha,\gamma}(u_1,u_0,u_4,u_3,u_2,v_1,v_0,v_4,v_3,v_2)$
(15) $M^1_{\beta,\alpha,\gamma}(u_0,u_4,u_3,u_2,u_1,v_0,v_4,v_3,v_2,v_1)$
(16) $M^2_{\beta,\alpha,\gamma}(u_0,u_1,u_2,u_3,u_4,v_0,v_1,v_2,v_3,v_4)$
(17) $M^2_{\beta,\alpha,\gamma}(u_1,u_2,u_3,u_4,u_0,v_1,v_2,v_3,v_4,v_0)$
(18) $M^2_{\beta,\alpha,\gamma}(u_2,u_3,u_4,u_0,u_1,v_2,v_3,v_4,v_0,v_1)$
(19) $M^2_{\beta,\alpha,\gamma}(u_3,u_4,u_0,u_1,u_2,v_3,v_4,v_0,v_1,v_2)$
(20) $M^2_{\beta,\alpha,\gamma}(u_4,u_0,u_1,u_2,u_3,v_4,v_0,v_1,v_2,v_3)$
(21) $M^2_{\beta,\alpha,\gamma}(u_4,u_3,u_2,u_1,u_0,v_4,v_3,v_2,v_1,v_0)$
(22) $M^2_{\beta,\alpha,\gamma}(u_3,u_2,u_1,u_0,u_4,v_3,v_2,v_1,v_0,v_4)$
(23) $M^2_{\beta,\alpha,\gamma}(u_2,u_1,u_0,u_4,u_3,v_2,v_1,v_0,v_4,v_3)$
(24) $M^2_{\beta,\alpha,\gamma}(u_1,u_0,u_4,u_3,u_2,v_1,v_0,v_4,v_3,v_2)$
(25) $M^2_{\beta,\alpha,\gamma}(u_0,u_4,u_3,u_2,u_1,v_0,v_4,v_3,v_2,v_1)$

Moreover, if the coloring is untwisted, then one of the cases (1), (2), (3), (4), or (5) must hold.

Remark Note that the question of which of the 25 cases is determined by a choice of $i,j\in\{0,1,2,3,4\}$ such that $u_i \stackrel{\gamma}{\rightarrow} \text{——}\, v_0,v_1,v_2,v_3,v_4$ and $v_j \stackrel{\gamma}{\rightarrow} \text{——}\, u_0,u_1,u_2,u_3,u_4$.

Proof. We omit the proof. □

Lemma 10. *Let* $u_0,u_1,u_2,u_3,u_4,v_0,v_1,v_2,v_3,v_4$ *be vertices in some good edge coloring of a complete graph on either* 15 *or* 16 *vertices, colored with the three pairwise distinct colors* α,β,γ*. Suppose that* *the following hold:*

$$\{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} = \emptyset$$
$$P_{\beta,\alpha}(u_3, u_4, u_0, u_1, u_2)$$
$$P_{\beta,\alpha}(v_3, v_4, v_0, v_1, v_2)$$
$$u_3 \overset{\gamma}{\to} \text{——} v_3$$
$$u_0 \overset{\gamma}{\to} \text{——} v_0$$
$$u_2 \overset{\gamma}{\to} \text{——} v_2$$

Then either both $u_1 \overset{\gamma}{\to} \text{——} v_1$ *and* $u_4 \overset{\gamma}{\to} \text{——} v_4$ *hold or else one of the following 10 cases must hold:*

(1) $M^1_{\beta,\alpha,\gamma}(u_4, u_0, u_1, u_2, u_3, v_2, v_3, v_4, v_1, v_0)$
(2) $M^1_{\beta,\alpha,\gamma}(u_1, u_0, u_4, u_3, u_2, v_3, v_2, v_1, v_0, v_4)$
(3) $M^0_{\beta,\alpha,\gamma}(u_1, u_2, u_3, u_4, u_0, v_4, v_3, v_2, v_1, v_0)$
(4) $M^0_{\beta,\alpha,\gamma}(u_0, u_1, u_2, u_3, u_4, v_0, v_4, v_3, v_2, v_1)$
(5) $M^2_{\beta,\alpha,\gamma}(u_3, u_4, u_0, u_1, u_2, v_2, v_1, v_0, v_4, v_3)$
(6) $M^2_{\beta,\alpha,\gamma}(u_2, u_1, u_0, u_4, u_3, v_3, v_4, v_0, v_1, v_2)$
(7) $M^2_{\beta,\alpha,\gamma}(u_1, u_0, u_4, u_3, u_2, v_0, v_1, v_2, v_3, v_4)$
(8) $M^2_{\beta,\alpha,\gamma}(u_4, u_0, u_1, u_2, u_3, v_0, v_4, v_3, v_2, v_1)$
(9) $M^2_{\beta,\alpha,\gamma}(u_4, u_3, u_2, u_1, u_0, v_2, v_3, v_4, v_0, v_1)$
(10) $M^2_{\beta,\alpha,\gamma}(u_1, u_2, u_3, u_4, u_0, v_3, v_2, v_1, v_0, v_4)$

Moreover, if the coloring is untwisted, then either both $u_1 \overset{\gamma}{\to} \text{——} v_1$ *and* $u_4 \overset{\gamma}{\to} \text{——} v_4$ *hold or else one of the cases (3) or (4) must hold.*

Remark Note that under the hypotheses of Lemma 10, in case we have both $u_1 \overset{\gamma}{\to} \text{——} v_1$ and $u_4 \overset{\gamma}{\to} \text{——} v_4$ holding, we see that the hypotheses of Lemma 9 are satisfied. Thus, Lemma 10 says that under the hypotheses of Lemma 10, either one of the 10 cases of Lemma 10 holds or else one of the 25 cases of Lemma 9 holds.

Proof. We omit the proof. □

Lemma 11. *Let* $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ *be vertices in some good edge coloring of a complete graph on either* 15 *or* 16 *vertices, colored with the three pairwise distinct colors* α, β, γ*. Suppose that the following hold:*

$$\{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} = \emptyset$$
$$P_{\beta,\alpha}(u_3, u_4, u_0, u_1, u_2)$$
$$P_{\beta,\alpha}(v_3, v_4, v_0, v_1, v_2)$$
$$u_4 \overset{\gamma}{\to} \text{——} v_4$$
$$u_0 \overset{\gamma}{\to} \text{——} v_0$$
$$u_1 \overset{\gamma}{\to} \text{——} v_1$$

Then either both $u_2 \overset{\gamma}{\to} \text{——} v_2$ *and* $u_3 \overset{\gamma}{\to} \text{——} v_3$ *hold or else one of the following 10 cases must hold:*

(1) $M^2_{\beta,\alpha,\gamma}(u_0, u_4, u_3, u_2, u_1, v_4, v_3, v_2, v_1, v_0)$
(2) $M^2_{\beta,\alpha,\gamma}(u_0, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_0)$
(3) $M^0_{\beta,\alpha,\gamma}(u_4, u_0, u_1, u_2, u_3, v_1, v_0, v_4, v_3, v_2)$

(4) $M^0_{\beta,\alpha,\gamma}(u_2,u_3,u_4,u_0,u_1,v_3,v_2,v_1,v_0,v_4)$
(5) $M^1_{\beta,\alpha,\gamma}(u_3,u_4,u_0,u_1,u_2,v_2,v_1,v_0,v_4,v_3)$
(6) $M^1_{\beta,\alpha,\gamma}(u_2,u_1,u_0,u_4,u_3,v_3,v_4,v_0,v_1,v_2)$
(7) $M^1_{\beta,\alpha,\gamma}(u_1,u_2,u_3,u_4,u_0,v_1,v_0,v_4,v_3,v_2)$
(8) $M^1_{\beta,\alpha,\gamma}(u_4,u_3,u_2,u_1,u_0,v_4,v_0,v_1,v_2,v_3)$
(9) $M^1_{\beta,\alpha,\gamma}(u_2,u_3,u_4,u_0,u_1,v_4,v_0,v_1,v_2,v_3)$
(10) $M^1_{\beta,\alpha,\gamma}(u_3,u_2,u_1,u_0,u_4,v_0,v_1,v_2,v_3,v_4)$

Moreover, if the coloring is untwisted, then either both $u_2 \stackrel{\gamma}{\to} \text{——}\, v_2$ *and* $u_3 \stackrel{\gamma}{\to} \text{——}\, v_3$ *hold or else one of the cases (3) or (4) must hold.*

Remark Note that under the hypotheses of Lemma 11, in case we have both $u_2 \stackrel{\gamma}{\to} \text{——}\, v_2$ and $u_3 \stackrel{\gamma}{\to} \text{——}\, v_3$ holding, we see that the hypotheses of Lemma A are satisfied. Thus, Lemma 11 says that under the hypotheses of Lemma 11, either one of the 10 cases of Lemma 11 holds or else one of the 25 cases of Lemma 9 holds.

Proof. We omit the proof. □

Lemma 12. *Let* $u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8$ *be vertices in some good edge coloring of a complete graph on either* 15 *or* 16 *vertices, colored with the three pairwise distinct colors* α,β,γ. *Suppose that the following hold:*

$$\{u_2,u_3,u_4\}\cap\{u_6,u_7,u_8\}=\emptyset$$
$$P_{\gamma,\beta}(u_1,u_2,u_3,u_4,u_5)$$
$$P_{\gamma,\beta}(u_5,u_6,u_7,u_8,u_1)$$

Then one of the following 5 cases must hold:

(1) $N^0_{\alpha,\beta,\gamma}(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)$
(2) $N^1_{\alpha,\beta,\gamma}(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)$
(3) $N^1_{\alpha,\beta,\gamma}(u_1,u_8,u_7,u_6,u_5,u_4,u_3,u_2)$
(4) $N^2_{\alpha,\beta,\gamma}(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)$
(5) $N^2_{\alpha,\beta,\gamma}(u_1,u_8,u_7,u_6,u_5,u_4,u_3,u_2)$

Moreover, if the coloring is untwisted, then case (1) must hold.

Proof. We omit the proof. □

Lemma 13. *Let* $u_0,u_1,u_2,u_3,u_4,v_0,v_1,v_2,v_3,v_4$ *be vertices in some good edge coloring of a complete graph on either* 15 *or* 16 *vertices, colored with the three pairwise distinct colors* α,β,γ. *Suppose that the following hold:*

$$\{u_0,u_1,u_2,u_3,u_4\}\cap\{v_0,v_1,v_2,v_3,v_4\}=\emptyset$$
$$P_{\beta,\alpha}(u_3,u_4,u_0,u_1,u_2)$$
$$P_{\beta,\alpha}(v_3,v_4,v_0,v_1,v_2)$$
$$u_0 \stackrel{\gamma}{\to} \text{——}\, v_0,v_1,v_2,v_3,v_4$$
$$v_0 \stackrel{\gamma}{\to} \text{——}\, u_0,u_1,u_2,u_3,u_4$$

Then one of the following 10 cases must hold:

(1) $M^0_{\beta,\alpha,\gamma}(u_3,u_4,u_0,u_1,u_2,v_3,v_4,v_0,v_1,v_2)$
(2) $M^0_{\beta,\alpha,\gamma}(u_3,u_4,u_0,u_1,u_2,v_2,v_1,v_0,v_4,v_3)$
(3) $M^1_{\beta,\alpha,\gamma}(u_2,u_3,u_4,u_0,u_1,v_4,v_0,v_1,v_2,v_3)$

(4) $M^1_{\beta,\alpha,\gamma}(u_2, u_3, u_4, u_0, u_1, v_1, v_0, v_4, v_3, v_2)$
(5) $M^1_{\beta,\alpha,\gamma}(u_3, u_2, u_1, u_0, u_4, v_4, v_0, v_1, v_2, v_3)$
(6) $M^1_{\beta,\alpha,\gamma}(u_3, u_2, u_1, u_0, u_4, v_1, v_0, v_4, v_3, v_2)$
(7) $M^2_{\beta,\alpha,\gamma}(u_1, u_2, u_3, u_4, u_0, v_0, v_1, v_2, v_3, v_4)$
(8) $M^2_{\beta,\alpha,\gamma}(u_1, u_2, u_3, u_4, u_0, v_0, v_4, v_3, v_2, v_1)$
(9) $M^2_{\beta,\alpha,\gamma}(u_4, u_3, u_2, u_1, u_0, v_0, v_1, v_2, v_3, v_4)$
(10) $M^2_{\beta,\alpha,\gamma}(u_4, u_3, u_2, u_1, u_0, v_0, v_4, v_3, v_2, v_1)$

Moreover, if the coloring is untwisted, then one of the cases (1) or (2) must hold.

Proof. We omit the proof. □

Lemma 14. *Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ. Suppose that the following hold:*

$$\{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} = \emptyset$$
$$P_{\beta,\alpha}(u_3, u_4, u_0, u_1, u_2)$$
$$P_{\beta,\alpha}(v_3, v_4, v_0, v_1, v_2)$$

Then there exists some $x \in \{u_0, u_1, u_2, u_3, u_4\}$ and some $y \in \{v_0, v_1, v_2, v_3, v_4\}$ such that

$$x \stackrel{\gamma}{\to} \text{——}\, v_0, v_1, v_2, v_3, v_4 \quad \text{and} \quad y \stackrel{\gamma}{\to} \text{——}\, u_0, u_1, u_2, u_3, u_4.$$

Furthermore, if $u_i \stackrel{\eta}{\to} \text{——}\, x$ for some $\eta \in \{\alpha, \beta\}$, then

$$\|\{j \in \{0,1,2,3,4\} \mid u_i \stackrel{\eta}{\to} \text{——}\, v_j\}\| = \|\{j \in \{0,1,2,3,4\} \mid u_i \stackrel{\gamma}{\to} \text{——}\, v_j\}\| = 2.$$

Similarly, if $v_i \stackrel{\eta}{\to} \text{——}\, y$ for some $\eta \in \{\alpha, \beta\}$, then

$$\|\{j \in \{0,1,2,3,4\} \mid v_i \stackrel{\eta}{\to} \text{——}\, u_j\}\| = \|\{j \in \{0,1,2,3,4\} \mid v_i \stackrel{\gamma}{\to} \text{——}\, u_j\}\| = 2.$$

Proof. We omit the proof. □

Lemma 15. *Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ. Suppose that the following hold:*

$$P_{\beta,\alpha}(u_3, u_4, u_0, u_1, u_2)$$
$$P_{\beta,\alpha}(v_3, v_4, v_0, v_1, v_2)$$

Then

$$\|\{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\}\| \in \{0, 2, 5\}.$$

Proof. We omit the proof. □

Lemma 16. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the colors α, β, and γ. Let $\|U\| \in \{15, 16\}$ and suppose that $A, B \subseteq U$ each have no edges of color γ in the induced colorings, where $\|A\| = \|B\| = 5$ and $A \cap B = \emptyset$. Then, there exist $x_0, x_1, x_2, x_3, x_4 \in A$ and $y_0, y_1, y_2, y_3, y_4 \in B$ such that $M^i_{\beta,\alpha,\gamma}(x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, y_3, y_4)$ holds for some $i \in \{0, 1, 2\}$.*

Proof. We omit the proof. □

Lemma 17. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the colors α, β, and γ. Let $\|U\| = 16$ and suppose that $a, b, c \in U$ with $a \stackrel{\gamma}{\to} \text{——}\, b \stackrel{\beta}{\to} \text{——}\, c \stackrel{\gamma}{\to} \text{——}\, a$. Then there exists a unique $d \in U$ such that $d \stackrel{\alpha}{\to} \text{——}\, a, b, c$.*

Proof. We omit the proof. □

4. The Local Arguments

4.1. **3.1. Attaching Sets of Cardinality 15 or 16.**

Theorem 1. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $u \neq v$ and $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \leq 14$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| \geq 15$. Since $S_\delta(u) \cap S_\delta(v) \neq \emptyset$, we must have $u \xrightarrow{\eta} \!\!—\!\!— v$ for some $\eta \in \{\alpha, \beta, \gamma\}$. Without loss of generality, we may assume that $u \xrightarrow{\gamma} \!\!—\!\!— v$. Thus, we have $S_\gamma(u) \cap S_\gamma(v) = \emptyset$, so that

$$S_\gamma(u) = \{v\} \uplus \big(S_\gamma(u) \cap S_\alpha(v)\big) \uplus \big(S_\gamma(u) \cap S_\beta(v)\big) \uplus \big(S_\gamma(u) \cap S_\delta(v)\big).$$

It is clear that

$$\big\|S_\gamma(u) \cap S_\alpha(v)\big\|, \big\|S_\gamma(u) \cap S_\beta(v)\big\| \leq R(3,3;2) - 1 = 6 - 1 = 5.$$

Also, we have $\|S_\gamma(u)\| \geq 13$ by Proposition 1, so that

$$13 \leq 1 + 5 + 5 + \big\|S_\gamma(u) \cap S_\delta(v)\big\|.$$

Thus, we have

$$\big\|S_\gamma(u) \cap S_\delta(v)\big\| \geq 2.$$

Now, note that

$$S_\delta(v) = \big(S_\alpha(u) \cap S_\delta(v)\big) \uplus \big(S_\beta(u) \cap S_\delta(v)\big) \uplus \big(S_\gamma(u) \cap S_\delta(v)\big) \uplus \big(S_\delta(u) \cap S_\delta(v)\big)$$

so that

$$16 \geq \big\|S_\alpha(u) \cap S_\delta(v)\big\| + \big\|S_\beta(u) \cap S_\delta(v)\big\| + 2 + 15,$$

which gives the desired contradiction. The theorem is proved. □

4.2. **3.2. Attaching Sets of Cardinality 14.**

Proposition 5. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 14$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [5,5,3]. \tag{4}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big), \tag{5}$$

so that

$$14 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| \\ + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|. \tag{6}$$

Thus, we have

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ \in [5, 5, 3] \cup [5, 4, 4]. \tag{7}$$

The proposition now follows by an application of Proposition 4. $\square$

Theorem 2. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 14$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 14$. Let $S_\delta(u) \sim S_\delta(v) = \{a, b\}$. Without loss of generality, we may assume that $a \stackrel{\gamma}{\longrightarrow} b$. By Lemma 6(2), there exists some $x \in S_\delta(u)$ with $a \stackrel{\alpha}{\longrightarrow} x \stackrel{\beta}{\longrightarrow} b$. Clearly, we have $x \in S_\delta(u) \cap S_\delta(v)$. Note that

$$\big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \uplus \{a\} = S_\delta(u) \cap S_\alpha(x),$$

$$\big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \{b\} = S_\delta(u) \cap S_\beta(x),$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = S_\delta(u) \cap S_\gamma(x).$$

But $\|S_\delta(u) \cap S_\eta(x)\| = 5$ for $\eta \in \{\alpha, \beta, \gamma\}$ by Lemma 4. Thus, we have

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ = (4, 4, 5) \notin [5, 5, 3], \tag{8}$$

which contradicts Proposition 5. The theorem is proved. $\square$

4.3. **3.3. Attaching Sets of Cardinality** 13.

Proposition 6. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 13$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ \in [5, 5, 2]. \tag{9}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \\ \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big), \tag{10}$$

so that

$$13 = \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \tag{11}$$

Thus, we have

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [5,5,2] \cup [5,4,3] \cup [4,4,4]. \tag{12}$$

The proposition now follows by an application of Proposition 4. □

Theorem 3. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 13$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 13$. Let $S_\delta(u) \sim S_\delta(v) = \{a, b, c\}$. For any $x \in S_\delta(u) \cap S_\delta(v)$, we must have $x \stackrel{\eta}{\to}\!\!\text{——}\, a, b, c$ for some $\eta \in \{\alpha, \beta, \gamma\}$, by Proposition 2 and Proposition 6. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \biguplus_{\eta \in \{\alpha,\beta,\gamma\}} \big(S_\delta(u) \cap S_\delta(v) \cap S_\eta(a) \cap S_\eta(b) \cap S_\eta(c)\big).$$

This will give the contradiction

$$13 \leq 1 + 1 + 1$$

once we show that

$$\|S_\delta(u) \cap S_\eta(a) \cap S_\eta(b) \cap S_\eta(c)\| \leq 1$$

for any $\eta \in \{\alpha, \beta, \gamma\}$. Suppose not. Then we may find distinct $x, y \in S_\delta(u) \cap S_\eta(a) \cap S_\eta(b) \cap S_\eta(c)$. But Lemma 6(1) implies that

$$\|\{\, z \in S_\delta(u) \mid x \stackrel{\eta}{\to}\!\!\text{——}\, z \stackrel{\eta}{\to}\!\!\text{——}\, y \,\}\| = 2,$$

which is impossible since $a, b, c \in \{\, z \in S_\delta(u) \mid x \stackrel{\eta}{\to}\!\!\text{——}\, z \stackrel{\eta}{\to}\!\!\text{——}\, y \,\}$. The theorem is proved. □

4.4. **3.4. Attaching Sets of Cardinality 12.**

Proposition 7. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 12$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [5,5,1] \cup [5,4,2]. \tag{13}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big), \tag{14}$$

so that

$$12 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|. \tag{15}$$

Thus, we have

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [5,5,1] \cup [5,4,2] \cup [5,3,3] \cup [4,4,3]. \tag{16}$$

The proposition now follows by an application of Proposition 4. ☐

Theorem 4. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 12$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 12$. Let $S_\delta(u) \sim (S_\delta(u) \cap S_\delta(v)) = \{a_0, a_1, a_2, a_3\}$. For each $i \in \{0, 1, 2, 3\}$, we define

$$U_i = \bigcup_{\eta \in \{\alpha,\beta,\gamma\}} \bigcap_{j \neq i} \big(S_\delta(u) \cap S_\eta(a_j)\big).$$

By Proposition 2 and Proposition 7, we see that

$$S_\delta(u) \cap S_\delta(v) \subseteq U_0 \cup U_1 \cup U_2 \cup U_3.$$

This will give the contradiction

$$12 \leq 1 + 1 + 1 + 1$$

once we show that $\|U_i\| \leq 1$ for each $i \in \{0, 1, 2, 3\}$. Suppose not. Then we may find distinct $x, y \in U_i$ for some fixed i. Without loss of generality, we may assume that $i = 3$. Then there exist $\eta, \xi \in \{\alpha, \beta, \gamma\}$ such that $x \overset{\eta}{\to} \!\!-\!\!-\!\!-\, a_0, a_1, a_2$ and $y \overset{\xi}{\to} \!\!-\!\!-\!\!-\, a_0, a_1, a_2$. But Lemma 6 implies that

$$\|\{\, z \in S_\delta(u) \mid x \overset{\eta}{\to} \!\!-\!\!-\!\!-\, z \overset{\xi}{\to} \!\!-\!\!-\!\!-\, y \,\}\| \leq 2,$$

which is impossible since $a_0, a_1, a_2 \in \{\, z \in S_\delta(u) \mid x \overset{\eta}{\to} \!\!-\!\!-\!\!-\, z \overset{\xi}{\to} \!\!-\!\!-\!\!-\, y \,\}$. The theorem is proved. ☐

4.5. **3.5. Attaching Sets of Cardinality** 11.

Proposition 8. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 11$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [5,5,0] \cup [5,3,2]. \tag{17}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big), \tag{18}$$

so that

$$11 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| \\ + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|. \tag{19}$$

Thus, we have

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ \in [5,5,0] \cup [5,4,1] \cup [5,3,2] \cup [4,4,2] \cup [4,3,3]. \tag{20}$$

The proposition now follows by an application of Proposition 4. □

Theorem 5. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 11$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 11$. For any $\eta \in \{\alpha, \beta, \gamma\}$ we define

$$U_\eta = \big\{\, x \in S_\delta(u) \cup S_\delta(v) \;\big|\; \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 5 \,\big\}.$$

By Proposition 8, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \cup U_\beta \cup U_\gamma,$$

so that we must have $\|U_\eta\| \geq 4$ for some η. Fix such an $\eta \in \{\alpha, \beta, \gamma\}$. We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges. The elements of U_η all have degree 5 in this graph.

Clearly, not all of the edges in U_η can be of color η. Choose distinct $x, y \in U_\eta$ such that the edge from x to y is not of color η. By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But, $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \subseteq S_\delta(v) = S_\delta(u) \cap S_\eta(x) \subseteq S_\delta(v)$, since $\|S_\delta(u) \cap S_\eta(x)\| = 5$ and $x \in U_\eta$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1, v_2, z\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0, w_1\}.$$

Clearly, we cannot have $u_i \overset{\eta}{\to}\!\!\text{——}\; w_j$ or $v_i \overset{\eta}{\to}\!\!\text{——}\; w_j$ for any $i \in \{0,1,2\}$ and $j \in \{0,1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $w_0 \overset{\eta}{\to}\!\!\text{——}\; w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) \subseteq \{x, y, z\}$ for all $i \in \{0,1\}$, so that $w_0, w_1 \notin U_\eta$. Now, since $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1, v_2, z\}$ with $\|U_\eta\| \geq 4$, we must have either $u_i \in U_\eta$ or $v_i \in U_\eta$ for some $i \in \{0,1,2\}$. Without loss of generality, we may assume that $u_0 \in U_\eta$. We cannot have $u_0 \overset{\eta}{\to}\!\!\text{——}\; u_i$ for any $i \in \{1,2\}$, since othewrwise we would have a monochromatic triangle. But then we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_0) \subseteq \{x, z, v_0, v_1, v_2\}$, so that, since $u_0 \in U_\eta$, we must have $u_0 \overset{\eta}{\to}\!\!\text{——}\; v_0, v_1, v_2$, which contradicts Lemma 6(1) since $y \overset{\eta}{\to}\!\!\text{——}\; v_0, v_1, v_2$. □

4.6. **3.6. Attaching sets of Cardinality 10.**

Proposition 9. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 10$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned}(21)\quad &\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in [5, 4, 0] \cup [5, 2, 2].\end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned}(22)\quad S_\delta(u) \cap S_\delta(v) = \{x\} &\uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \\ &\uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big),\end{aligned}$$

so that

$$\begin{aligned}(23)\quad 10 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 &+ \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| \\ &+ \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}(24)\quad &\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ &\qquad\qquad\qquad \in [5, 4, 0] \cup [5, 3, 1] \cup [5, 2, 2] \cup [4, 4, 2] \cup [4, 3, 3].\end{aligned}$$

The proposition now follows by an application of Proposition 4. ☐

Theorem 6. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 10$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 10$. For any $\eta \in \{\alpha, \beta, \gamma\}$ we define

$$U_\eta = \big\{\, x \in S_\delta(u) \cup S_\delta(v) \;\big|\; \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 5 \,\big\}.$$

By Proposition 9, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \cup U_\beta \cup U_\gamma,$$

so that we must have $\|U_\eta\| \geq 4$ for some η. Fix such an $\eta \in \{\alpha, \beta, \gamma\}$. We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges. The elements of U_η all have degree 5 in this graph.

Clearly, not all of the edges in U_η can be of color η. Choose distinct $x, y \in U_\eta$ such that the edge from x to y is not of color η. By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But, $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \subseteq S_\delta(v) = S_\delta(u) \cap S_\eta(x) \subseteq S_\delta(v)$, since $\|S_\delta(u) \cap S_\eta(x)\| = 5$ and $x \in U_\eta$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1, v_2\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0, w_1\}.$$

Clearly, we cannot have $u_i \overset{\eta}{\to} \text{——} w_j$ or $v_i \overset{\eta}{\to} \text{——} w_j$ for any $i \in \{0,1,2\}$ and $j \in \{0,1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $w_0 \overset{\eta}{\to} \text{——} w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) = \{x, y\}$ for all $i \in \{0,1\}$, so that $w_0, w_1 \notin U_\eta$. Now, since $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1, v_2\}$ with $\|U_\eta\| \geq 4$, we must have either $u_i \in U_\eta$ or $v_i \in U_\eta$ for some $i \in \{0,1,2\}$. Without loss of generality, we may assume that $u_0 \in U_\eta$. we cannot have $u_0 \overset{\eta}{\to} \text{——} u_i$ for any $i \in \{1,2\}$, since otherwise we would have a monochromatic triangle. But now we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_0) \subseteq \{x, v_0, v_1, v_2\}$, which contradicts the fact that $u_0 \in U_\eta$. □

4.7. **3.7. Attaching Sets of Cardinality** 9**.**

Proposition 10. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 9$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned}(25)\quad &\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ &\qquad \in [5,3,0] \cup [5,2,1] \cup [4,2,2].\end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned}(26)\quad S_\delta(u) \cap S_\delta(v) = \{x\} &\uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \\ &\uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big),\end{aligned}$$

so that

$$\begin{aligned}(27)\quad 9 = \big\|S_\delta(u) \cap S_\delta(v)\big\| &= 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| \\ &+ \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}(28)\quad &\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ &\qquad \in [5,3,0] \cup [5,2,1] \cup [4,4,0] \cup [4,3,1] \cup [4,2,2] \cup [3,3,2].\end{aligned}$$

The proposition now follows by an application of Proposition 4. □

Theorem 7. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 9$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 9$. For any $\eta \in \{\alpha, \beta, \gamma\}$ we define

$$U_\eta = \left\{ x \in S_\delta(u) \cup S_\delta(v) \;\middle|\; \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \geq 4 \right\}$$

and

$$U'_\eta = \left\{ x \in S_\delta(u) \cup S_\delta(v) \;\middle|\; \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 5 \right\}$$

By Proposition 10, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \cup U_\beta \cup U_\gamma,$$

so that we must have $\|U_\eta\| \geq 3$ for some η. Fix such an $\eta \in \{\alpha, \beta, \gamma\}$. We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges. The elements of U_η all have degree either 4 or 5 and the elements of $U'_\eta \subseteq U_\eta$ all have degree 5 in this graph.

Clearly, not all of the edges in U_η can be of color η. Choose distinct $x, y \in U_\eta$ such that the edge from x to y is not of color η.

First, we show that either $x \notin U'_\eta$ or $y \notin U'_\eta$. Suppose not. Then $x, y \in U'_\eta$. By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But, $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = S_\delta(u) \cap S_\eta(x)$, since $\|S_\delta(u) \cap S_\eta(x)\| = 5$ by Proposition 2 and $x \in U'_\eta$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

Thus, we may suppose that there are pairwise distinct

$$x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1, v_2 \in S_\delta(u) \cap S_\delta(v),$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0, w_1\}.$$

But this is clearly impossible, since $S_\delta(u) \cap S_\delta(v)$ has only 9 elements. Thus, we have shown that either $x \notin U'_\eta$ or $y \notin U'_\eta$, as desired.

Next, we show that $x, y \notin U'_\eta$. Suppose not. Then we may assume, without loss of generality, that $x \in U'_\eta$ and $y \notin U'_\eta$. By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But, $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = S_\delta(u) \cap S_\eta(x)$, since $\|S_\delta(u) \cap S_\eta(x)\| = 5$ by Proposition 2 and $x \in U'_\eta$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, w_0, w_1\}.$$

Clearly, we cannot have $u_i \overset{\eta}{\to} \!\!\!\!\!-\!\!\!- w_j$ for any $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $v_i \overset{\eta}{\to} \!\!\!\!\!-\!\!\!- w_j$ for any $i \in \{0, 1\}$ and $j \in \{0, 1\}$, since we would then have a monochromatic triangle. Likewise, we cannot have $w_0 \overset{\eta}{\to} \!\!\!\!\!-\!\!\!- w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) = \{x, y\}$ for all $i \in \{0, 1\}$, so that $w_0, w_1 \notin U_\eta$. Now, since $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1\}$ with $\|U_\eta\| \geq 3$, we must have either $u_i \in U_\eta$ for some $i \in \{0, 1, 2\}$ or $v_i \in U_\eta$ for some $i \in \{0, 1\}$. But, for any

$i \in \{0,1,2\}$, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_i) \subseteq \{x, v_0, v_1\}$, so that $u_i \notin U_\eta$. Thus, we must have $v_i \in U_\eta$ for some $i \in \{0,1\}$. Without loss of generality, we may assume that $v_0 \in U_\eta$. But then we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(v_0) \subseteq \{y, u_0, u_1, u_2\}$ so that, since $v_0 \in U_\eta$, we must have $v_0 \overset{\eta}{\to} \!\!\text{——}\, u_0, u_1, u_2$, which contradicts Lemma 6(1) since $x \overset{\eta}{\to} \!\!\text{——}\, u_0, u_1, u_2$. Thus, we have shown that $x, y \notin U'_\eta$, as desired.

Now, by Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But, $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = \|S_\delta(u) \cap S_\eta(x)\| - 1$, since $\|S_\delta(u) \cap S_\eta(x)\| = 5$ by Proposition 2 and $y \in U_\eta \sim U'_\eta$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| \in \{1, 2\}.$$

Next, we show that $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2$. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 1$, so that we may assume that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, v_0, v_1, v_2\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0\}.$$

We cannot have $w_0 \overset{\eta}{\to} \!\!\text{——}\, u_i$ or $w_0 \overset{\eta}{\to} \!\!\text{——}\, v_i$ for any $i \in \{0,1,2\}$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_0) = \{x, y\}$, so that $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1, v_2\}$. Since $\|U_\eta\| \geq 3$, we have either $u_i \in U_\eta$ or $v_i \in U_\eta$ for some $i \in \{0,1,2\}$. without loss of generality, we may suppose that $u_0 \in U_\eta$. We cannot have $u_0 \overset{\eta}{\to} \!\!\text{——}\, u_i$ for any $i \in \{1,2\}$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_0) \subseteq \{x, v_0, v_1, v_2\}$, which implies that $u_0 \overset{\eta}{\to} \!\!\text{——}\, v_0, v_1, v_2$ since $u_0 \in U_\eta$. But this is impossible, by Lemma 6(1), since $y \overset{\eta}{\to} \!\!\text{——}\, v_0, v_1, v_2$. Thus, we have shown that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2,$$

as desired

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, w_0, w_1, v_0, v_1, z\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, w_0, w_1\}.$$

Clearly, we cannot have $u_i \overset{\eta}{\to} \!\!\text{——}\, w_j$ for any $i \in \{0,1\}$ and $j \in \{0,1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $v_i \overset{\eta}{\to} \!\!\text{——}\, w_j$ for any $i \in \{0,1\}$ and $j \in \{0,1\}$, since we would then have a monochromatic triangle. Likewise, we cannot have $w_0 \overset{\eta}{\to} \!\!\text{——}\, w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) \subseteq \{x, y, z\}$ for all $i \in \{0,1\}$, so that $w_0, w_1 \notin U_\eta$. Since $x \overset{\eta}{\to} \!\!\text{——}\, w_0 \overset{\eta}{\to} \!\!\text{——}\, y \overset{\eta}{\to} \!\!\text{——}\, w_1 \overset{\eta}{\to} \!\!\text{——}\, x$, we see by Lemma 8 that either $z \overset{\eta}{\to} \!\!\text{——}\, w_0$ or $z \overset{\eta}{\to} \!\!\text{——}\, w_1$ but not both. Without loss of generality, we may suppose that $z \overset{\eta}{\to} \!\!\text{——}\, w_0$. Thus, we have

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_0) = \{x, y, z\},$$

so that, by Proposition 10, we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_0)\| = 5$$

for some $\xi \in \{\alpha, \beta, \gamma\} \sim \{\eta\}$. But, $S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_0) \subseteq \{x, y, w_1, v_0, v_1\}$, so that we must have

$$w_1, v_0, v_1 \in S_\delta(u) \cap S_\eta(y) \cap S_\xi(w_0),$$

which is impossible, since

$$\|S_\delta(u) \cap S_\eta(y) \cap S_\xi(w_0)\| \leq R(3; 2) - 1 = 3 - 1 = 2.$$

The proof is complete. □

4.8. **3.8. Attaching Sets of Cardinality** 8**.**

Proposition 11. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 8$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned}(29) \quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big)& \\ \in [5, 2, 0] \cup [3, 2, 2].&\end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned}(30) \quad S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big)& \\ \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big),&\end{aligned}$$

so that

$$\begin{aligned}(31) \quad 8 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\|& \\ + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|.&\end{aligned}$$

Thus, we have

$$\begin{aligned}(32) \quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big)& \\ \in [5, 2, 0] \cup [5, 1, 1] \cup [4, 3, 0] \cup [4, 2, 1] \cup [3, 3, 1] \cup [3, 2, 2].&\end{aligned}$$

The proposition now follows by an application of Proposition 4. □

Proposition 12. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 8$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned}(33) \quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big)& \\ \in [3, 2, 2].&\end{aligned}$$

Proof. Suppose not. Then, by Proposition 11, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 5,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 2,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

Let $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{w_0, w_1, w_2, w_3, w_4\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{y_0, y_1\}$.

For each $i \in \{0,1,2,3,4\}$, we define

$$n_i = \big\|\{\, j \in \{0,1\} \mid w_i \overset{\alpha}{\to} \!\!-\!\!-\!\!- y_j \,\}\big\|.$$

For each $j \in \{0,1\}$, we define

$$m_j = \big\|\{\, i \in \{0,1,2,3,4\} \mid w_i \overset{\alpha}{\to} \!\!-\!\!-\!\!- y_j \,\}\big\|.$$

Note that we cannot have $w_i \overset{\alpha}{\to} \!\!-\!\!-\!\!- w_{i'}$ for any distinct $i, i' \in \{0,1,2,3,4\}$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(w_i) \subseteq \{x, y_0, y_1\}$ for any $i \in \{0,1,2,3,4\}$. By Proposition 11, we see that, for any $i \in \{0,1,2,3,4\}$, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(w_i)\| \in \{0,2,3,5\}$, so that $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(w_i)\| \geq 2$, since $w_i \overset{\alpha}{\to} \!\!-\!\!-\!\!- x$. Thus, for any $i \in \{0,1,2,3,4\}$, we have $w_i \overset{\alpha}{\to} \!\!-\!\!-\!\!- y_j$ for some $j \in \{0,1\}$, so that $n_i \geq 1$.

Thus, we have

$$m_0 + m_1 = n_0 + n_1 + n_2 + n_3 + n_4 \geq 1 + 1 + 1 + 1 + 1 = 5,$$

so that we must have $m_j \geq 3$ for some $j \in \{0,1\}$. Without loss of generality, we may suppose that $y_0 \overset{\alpha}{\to} \!\!-\!\!-\!\!- w_0, w_1, w_2$. But this is impossible, by Lemma 6(1), since $x \overset{\alpha}{\to} \!\!-\!\!-\!\!- w_0, w_1, w_2$. The proposition follows. □

Theorem 8. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 8$.*

Proof. By Proposition 12, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{2,3\}$ for any $x \in S_\delta(u) \cap S_\delta(v)$ and any $\eta \in \{\alpha, \beta, \gamma\}$. For each $\eta \in \{\alpha, \beta, \gamma\}$, we define

$$U_\eta = \big\{\, x \in S_\delta(u) \cap S_\delta(v) \;\big|\; \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 3 \,\big\}.$$

By Proposition 12, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \uplus U_\beta \uplus U_\gamma.$$

Note that the number of edges of color $\eta \in \{\alpha, \beta, \gamma\}$ in $S_\delta(u) \cap S_\delta(v)$ is

$$\frac{3 \cdot \|U_\eta\| + 2 \cdot (8 - \|U_\eta\|)}{2}.$$

It follows that $\|U_\eta\|$ is even for every $\eta \in \{\alpha, \beta, \gamma\}$. Thus, we have

$$\big(\|U_\alpha\|, \|U_\beta\|, \|U_\gamma\|\big) \in [8,0,0] \cup [6,2,0] \cup [4,4,0] \cup [4,2,2].$$

Thus, there exists some $\eta \in \{\alpha, \beta, \gamma\}$ such that

$$\|U_\eta\| \in \{0,2\}.$$

Fix such an η. We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges.

If $\|U_\eta\| = 0$, then the graph under consideration is a triangle free graph with 8 vertices which all have degree 2. Such a graph has exactly 8 edges. There are two such possibilities. The first possibility is

$$w_0 \overset{\eta}{\to} \relbar\relbar\, w_1 \overset{\eta}{\to} \relbar\relbar\, w_2 \overset{\eta}{\to} \relbar\relbar\, w_3 \overset{\eta}{\to} \relbar\relbar\, w_4 \overset{\eta}{\to} \relbar\relbar\, w_5 \overset{\eta}{\to} \relbar\relbar\, w_6 \overset{\eta}{\to} \relbar\relbar\, w_7 \overset{\eta}{\to} \relbar\relbar\, w_0$$

where $S_\delta(u) \cap S_\delta(v) = \{w_0, \ldots, w_7\}$, which fails by Lemma 7 by consideration of $w_0 \overset{\eta}{\to} \relbar\relbar\, w_1 \overset{\eta}{\to} \relbar\relbar\, w_2$ and $w_4 \overset{\eta}{\to} \relbar\relbar\, w_5$. The second possibility is

$$x_0 \overset{\eta}{\to} \relbar\relbar\, x_1 \overset{\eta}{\to} \relbar\relbar\, x_2 \overset{\eta}{\to} \relbar\relbar\, x_3 \overset{\eta}{\to} \relbar\relbar\, x_0$$

and

$$y_0 \overset{\eta}{\to} \relbar\relbar\, y_1 \overset{\eta}{\to} \relbar\relbar\, y_2 \overset{\eta}{\to} \relbar\relbar\, y_3 \overset{\eta}{\to} \relbar\relbar\, y_0,$$

where $S_\delta(u) \cap S_\delta(v) = \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\}$, which fails by Lemma 7 by consideration of $x_0 \overset{\eta}{\to} \relbar\relbar\, x_1 \overset{\eta}{\to} \relbar\relbar\, x_2$ and $y_0 \overset{\eta}{\to} \relbar\relbar\, y_1$. Thus, we cannot have $\|U_\eta\| = 0$.

Therefore, we must have $\|U_\eta\| = 2$. Thus,the graph under consideration is a triangle free graph with 8 vertices, 2 of which have degree 3, and 6 of which have degree 2. Such a graph has exactly 9 edges. Let $U_\eta = \{a, b\}$.

If $a \overset{\eta}{\to} \relbar\relbar\, b$, there are two possibilities for the remaining 8 edges. The first possibility is

$$a \overset{\eta}{\to} \relbar\relbar\, x_0 \overset{\eta}{\to} \relbar\relbar\, x_1 \overset{\eta}{\to} \relbar\relbar\, b$$

and

$$a \overset{\eta}{\to} \relbar\relbar\, y_0 \overset{\eta}{\to} \relbar\relbar\, y_1 \overset{\eta}{\to} \relbar\relbar\, y_2 \overset{\eta}{\to} \relbar\relbar\, y_3 \overset{\eta}{\to} \relbar\relbar\, b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, x_1, y_0, y_1, y_2, y_3\}$, which fails by Lemma 7, by consideration of $a \overset{\eta}{\to} \relbar\relbar\, x_0 \overset{\eta}{\to} \relbar\relbar\, x_1$ and $y_1 \overset{\eta}{\to} \relbar\relbar\, y_2$. The second possibility is

$$a \overset{\eta}{\to} \relbar\relbar\, x_0 \overset{\eta}{\to} \relbar\relbar\, x_1 \overset{\eta}{\to} \relbar\relbar\, x_2 \overset{\eta}{\to} \relbar\relbar\, b$$

and

$$a \overset{\eta}{\to} \relbar\relbar\, y_0 \overset{\eta}{\to} \relbar\relbar\, y_1 \overset{\eta}{\to} \relbar\relbar\, y_2 \overset{\eta}{\to} \relbar\relbar\, b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, x_1, x_2, y_0, y_1, y_2\}$, which fails by Lemma 7 by consideration of $x_0 \overset{\eta}{\to} \relbar\relbar\, x_1 \overset{\eta}{\to} \relbar\relbar\, x_2$ and $y_0 \overset{\eta}{\to} \relbar\relbar\, y_1$.

Thus, we cannot have $a \overset{\eta}{\to} \relbar\relbar\, b$. Here there are three possibilities. The first possibility is

$$a \overset{\eta}{\to} \relbar\relbar\, x_0 \overset{\eta}{\to} \relbar\relbar\, b$$

and

$$a \overset{\eta}{\to} \relbar\relbar\, y_0 \overset{\eta}{\to} \relbar\relbar\, y_1 \overset{\eta}{\to} \relbar\relbar\, b$$

and

$$a \overset{\eta}{\to} \relbar\relbar\, z_0 \overset{\eta}{\to} \relbar\relbar\, z_1 \overset{\eta}{\to} \relbar\relbar\, z_2 \overset{\eta}{\to} \relbar\relbar\, b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, y_0, y_1, z_0, z_1, z_2\}$, which fails by Lemma 7 by consideration of $a \overset{\eta}{\to} \relbar\relbar\, y_0 \overset{\eta}{\to} \relbar\relbar\, y_1$ and $z_1 \overset{\eta}{\to} \relbar\relbar\, z_2$. The second possibility is

$$a \overset{\eta}{\to} \relbar\relbar\, x_0 \overset{\eta}{\to} \relbar\relbar\, b$$

and

$$a \overset{\eta}{\to} \relbar\relbar\, y_0 \overset{\eta}{\to} \relbar\relbar\, b$$

and

$$a \overset{\eta}{\to} \relbar\relbar\, z_0 \overset{\eta}{\to} \relbar\relbar\, z_1 \overset{\eta}{\to} \relbar\relbar\, z_2 \overset{\eta}{\to} \relbar\relbar\, z_3 \overset{\eta}{\to} \relbar\relbar\, b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, y_0, z_0, z_1, z_2, z_3\}$, which fails by Lemma 7 by consideration of $z_0 \stackrel{\eta}{\to} \text{——}\, z_1 \stackrel{\eta}{\to} \text{——}\, z_2$ and $x_0 \stackrel{\eta}{\to} \text{——}\, b$. The third possibility is

$$a \stackrel{\eta}{\to} \text{——}\, x_0 \stackrel{\eta}{\to} \text{——}\, x_1 \stackrel{\eta}{\to} \text{——}\, b$$

and

$$a \stackrel{\eta}{\to} \text{——}\, y_0 \stackrel{\eta}{\to} \text{——}\, y_1 \stackrel{\eta}{\to} \text{——}\, b$$

and

$$a \stackrel{\eta}{\to} \text{——}\, z_0 \stackrel{\eta}{\to} \text{——}\, z_1 \stackrel{\eta}{\to} \text{——}\, b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, x_1, y_0, y_1, z_0, z_1\}$, which fails by Lemma 7 by consideration of $x_0 \stackrel{\eta}{\to} \text{——}\, a \stackrel{\eta}{\to} \text{——}\, y_0$ and $z_1 \stackrel{\eta}{\to} \text{——}\, b$.

The proof is complete ☐

4.9. **3.9. Attaching Sets of Cardinality 7.**

Proposition 13. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 7$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [5,1,0] \cup [4,2,0] \cup [2,2,2]. \tag{34}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big), \tag{35}$$

so that

$$7 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|. \tag{36}$$

Thus, we have

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [5,1,0] \cup [4,2,0] \cup [4,1,1] \cup [3,3,0] \cup [3,2,1] \cup [2,2,2]. \tag{37}$$

The proposition now follows by an application of Proposition 4. ☐

Proposition 14. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 7$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [4,2,0] \cup [2,2,2]. \tag{38}$$

Proof. Suppose not. Then, by Proposition 13, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 5,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 1,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

Let $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{w_0, w_1, w_2, w_3, w_4\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{y\}$.

By Lemma 6(3), we see that

$$\|S_\delta(u) \cap S_\alpha(x) \cap S_\eta(w_i)\| = 2$$

for any $i \in \{0, 1, 2, 3, 4\}$ and $\eta \in \{\beta, \alpha\}$x. But $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = S_\delta(u) \cap S_\alpha(x)$ since $\|S_\delta(u) \cap S_\alpha(x)\| = 5$ by Proposition 2 and $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 5$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) \cap S_\eta(w_i)\| = 2$$

for any $i \in \{0, 1, 2, 3, 4\}$ and $\eta \in \{\beta, \gamma\}$. Since $x \overset{\alpha}{\to} \text{——}\, w_i$ for all $i \in \{0, 1, 2, 3, 4\}$, we must have $w_i \overset{\alpha}{\to} \text{——}\, y$, by Proposition 13. But now we have $y \overset{\alpha}{\to} \text{——}\, w_0, w_1, w_2, w_3, w_4$, which is impossible by Lemma 6(1), since $x \overset{\alpha}{\to} \text{——}\, w_0, w_1, w_2, w_3, w_4$.

The proof is complete ☐

Proposition 15. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 7$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [2, 2, 2]. \tag{39}$$

Proof. Suppose not. Then, by Proposition 14, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 4,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 2,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

Let $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{w_0, w_1, w_2, w_3\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{y_0, y_1\}$.

By Lemma 6(3), we see that

$$\|S_\delta(u) \cap S_\alpha(x) \cap S_\eta(w_i)\| = 2$$

for any $i \in \{0, 1, 2, 3\}$ and $\eta \in \{\beta, \alpha\}$.

But $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = \|S_\delta(u) \cap S_\alpha(x)\| - 1$ since $\|S_\delta(u) \cap S_\alpha(x)\| = 5$ by Proposition 2 and $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 4$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) \cap S_\eta(w_i)\| \in \{1, 2\}$$

for any $i \in \{0, 1, 2, 3\}$ and $\eta \in \{\beta, \gamma\}$. Since $x \overset{\alpha}{\to} \text{——}\, w_i$ for all $i \in \{0, 1, 2, 3\}$, we must have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_i)\| \neq 0$$

for all $i \in \{0, 1, 2, 3\}$ and all $\xi \in \{\alpha, \beta, \gamma\}$. Thus, by Proposition 14, we see that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_i)\| = 2$$

for all $i \in \{0, 1, 2, 3\}$ and all $\xi \in \{\alpha, \beta, \gamma\}$.

Now, note that we cannot have $y_0 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- y_1$, since then we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i) \subseteq \{w_0, w_1, w_2, w_3, x\}$$

for all $i \in \{0,1\}$. Now, suppose that $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i)\| \neq 2$ for some $i \in \{0,1\}$. Then, since $y_i \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- x$, we must have, by Proposition 14, $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i)\| = 4$ But then $S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i) \cap S_\alpha(x) = 3$. But, this is impossible, since $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i) \cap S_\alpha(x)\| \leq R(3;2) - 1 = 3 - 1 = 2$. Thus, we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i)\| = 2.$$

In fact, we have now shown that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(z)\| = 2$$

for all $z \in S_\delta(u) \cap S_\delta(v)$.

Now, we consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the β colored edges. This graph is a triangle free graph on 7 vertices, all of which have degree 7. Such a graph has exactly 7 edges. There is only one possibility. We must have

$$z_0 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_1 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_2 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_3 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_4 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_5 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_6 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_0,$$

where $S_\delta(u) \cap S_\delta(v) = \{z_0 \ldots, z_6\}$, which fails, by Lemma 7, by consideration of $z_0 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_1 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_2$ and $z_4 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_5$. The proof is complete. □

Theorem 9. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 7$.*

Proof. Note that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(z)\| = 2$$

for all $z \in S_\delta(u) \cap S_\delta(v)$, by Proposition 15. Consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the β colored edges. This graph is a triangle free graph on 7 vertices, all of which have degree 2. Such a graph has exactly 7 edges. There is only one possibility. We must have

$$z_0 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_1 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_2 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_3 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_4 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_5 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_6 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_0,$$

where $S_\delta(u) \cap S_\delta(v) = \{z_0 \ldots, z_6\}$, which fails, by Lemma 7, by consideration of $z_0 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_1 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_2$ and $z_4 \overset{\beta}{\to} \!\!\!-\!\!\!-\!\!\!- z_5$.

The proof is complete. □

4.10. **3.10. Attaching Sets of Cardinality** 6.

Proposition 16. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 6$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned}(40)\quad &\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \\ &\qquad\in [5,0,0] \cup [3,2,0] \cup [2,2,1].\end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned}(41)\quad S_\delta(u)\cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\big)\\ \uplus \big(S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\big) \uplus \big(S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\big),\end{aligned}$$

so that

$$\begin{aligned}(42)\quad 6 = \big\|S_\delta(u)\cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\big\|\\ + \big\|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\big\| + \big\|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\big\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}(43)\quad \big(\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\|\big)\\ \in [5,0,0]\cup[4,1,0]\cup[3,2,0]\cup[2,2,1].\end{aligned}$$

The proposition now follows by an application of Proposition 4. □

Proposition 17. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u)\cap S_\delta(v)\| = 6$. If $x \in S_\delta(u)\cap S_\delta(v)$, then*

$$\begin{aligned}(44)\quad \big(\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\|\big)\\ \in [3,2,0]\cup[2,2,1].\end{aligned}$$

Proof. Suppose not. Then, by Proposition 16, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u)\cap S_\delta(v)$ such that

$$\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\| = 5,$$
$$\|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\| = 0,$$

and

$$\|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\| = 0.$$

By Proposition 1, either $\|S_\beta(x)\| \geq 15$ or $\|S_\gamma(x)\| \geq 15$. Without loss of generality, we may suppose that $\|S_\gamma(x)\| \geq 15$. By Proposition 2, we have $\|S_\delta(u)\cap S_\gamma(x)\| = 5$ and $\|S_\delta(v)\cap S_\gamma(x)\| = 5$. Thus, by Lemma 16, we see that there exist $x_0, x_1, x_2, x_3, x_4 \in S_\delta(u)\cap S_\gamma(x)$ and $y_0, y_1, y_2, y_3, y_4 \in S_\delta(v)\cap S_\gamma(x)$ such that

$$M^i_{\beta,\alpha,\delta}(x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, y_3, y_4)$$

for some $i \in \{0,1,2\}$. Note also that $S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x) = S_\delta(u)\cap S_\alpha(x)$, since $\|S_\delta(u)\cap S_\alpha(x)\| = 5$ by Proposition 2 and $\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\| = 5$.

First, suppose that $i = 0$. Note that $x \overset{\gamma}{\to}\!\relbar\!\relbar\, x_0 \overset{\beta}{\to}\!\relbar\!\relbar\, x_4 \overset{\gamma}{\to}\!\relbar\!\relbar\, x$ with $x, x_0, x_4 \in S_\delta(u)$. Thus, by Lemma 17, there exists some $w \in S_\delta(u)$ with $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x, x_0, x_4$. Since $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x_0 \overset{\alpha}{\to}\!\relbar\!\relbar\, y_3, y_4$ and $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x_4 \overset{\alpha}{\to}\!\relbar\!\relbar\, y_0, y_1$, we have $S_\delta(v)\cap S_\gamma(x)\cap S_\alpha(w) \subseteq \{y_2\}$. But, since $w \in S_\delta(u)\cap S_\alpha(x) \subseteq S_\delta(v)$, we see by Lemma 6(3), that $\|S_\delta(v)\cap S_\gamma(x)\cap S_\alpha(w)\| \geq 2$, which is impossible. Thus, $i \neq 0$.

Next, suppose that $i = 1$. Note that $x \overset{\gamma}{\to}\!\relbar\!\relbar\, x_0 \overset{\beta}{\to}\!\relbar\!\relbar\, x_1 \overset{\gamma}{\to}\!\relbar\!\relbar\, x$ with $x, x_0, x_1 \in S_\delta(u)$. Thus, by Lemma 17, there exists some $w \in S_\delta(u)$ with $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x, x_0, x_1$. Since $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x_0 \overset{\alpha}{\to}\!\relbar\!\relbar\, y_2, y_3$ and $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x_1 \overset{\alpha}{\to}\!\relbar\!\relbar\, y_0, y_4$, we have $S_\delta(v)\cap S_\gamma(x)\cap S_\alpha(w) \subseteq \{y_1\}$. But, since $w \in S_\delta(u)\cap S_\alpha(x) \subseteq S_\delta(v)$, we see by Lemma 6(3), that $\|S_\delta(v)\cap S_\gamma(x)\cap S_\alpha(w)\| \geq 2$, which is impossible. Thus, $i \neq 1$.

First, suppose that $i = 2$. Note that $x \overset{\gamma}{\to}\!\relbar\!\relbar\, x_2 \overset{\beta}{\to}\!\relbar\!\relbar\, x_3 \overset{\gamma}{\to}\!\relbar\!\relbar\, x$ with $x, x_2, x_3 \in S_\delta(u)$. Thus, by Lemma 17, there exists some $w \in S_\delta(u)$ with $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x, x_2, x_3$. Since $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x_2 \overset{\alpha}{\to}\!\relbar\!\relbar\, y_3, y_4$ and $w \overset{\alpha}{\to}\!\relbar\!\relbar\, x_3 \overset{\alpha}{\to}\!\relbar\!\relbar\, y_1$, we have $S_\delta(v)\cap S_\gamma(x)\cap S_\alpha(w) \subseteq \{y_0, y_2\}$. But, since

$w \in S_\delta(u) \cap S_\alpha(x) \subseteq S_\delta(v)$, we see by Lemma 6(3), that $\|S_\delta(v) \cap S_\gamma(x) \cap S_\alpha(w)\| \geq 2$, which implies that $w \overset{\alpha}{\to} —— y_0, y_2$. But this gives a monochromatic triangle, since $y_0 \overset{\alpha}{\to} —— y_2$. Thus, $i \neq 2$.

The proof is complete. □

Proposition 18. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose that $a, b, c, d \in S_\delta(u) \cap S_\delta(v)$. $a \overset{\alpha}{\to} —— b \overset{\alpha}{\to} —— c \overset{\alpha}{\to} —— d \overset{\alpha}{\to} —— a$ and $a \overset{\beta}{\to} —— c$ and $b \overset{\beta}{\to} —— d$. Then $\|S_\delta(u) \cap S_\delta(v)\| \notin \{4, 6\}$*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| \in \{4, 6\}$.

First, we show that there exists some $x \in \{a, b, c, d\}$ with $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 2$. Suppose not. Then there exist $a', b', c', d' \in S_\delta(u) \cap S_\delta(v) \sim \{a, b, c, d\}$ such that $a \overset{\alpha}{\to} —— a'$, $b \overset{\alpha}{\to} —— b'$, $c \overset{\alpha}{\to} —— c'$, and $d \overset{\alpha}{\to} —— d'$. By Lemma 8, we have $\|\{a, b, c, d, a', b', c', d'\}\| = 8$, which contradicts the fact that $\|S_\delta(u) \cap S_\delta(v)\| \in \{4, 6\}$. Thus, there exista some $x \in \{a, b, c, d\}$ with $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 2$, as desired.

Without loss of generality, we may suppose that $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(a)\| = 2$. By Lemma 4, there exist $x_0, x_1, x_2 \in S_\delta(u) \cap S_\alpha(a)$ and $y_0, y_1, y_2 \in S_\delta(v) \cap S_\alpha(a)$ such that both $P_{\beta,\gamma}(b, x_0, x_1, x_2, d)$ and $P_{\beta,\gamma}(b, y_0, y_1, y_2, d)$ hold.

Now, we show that $c \overset{\beta}{\to} —— x_0, x_2, y_0, y_2$. Suppose not. Without loss of generality, we may suppose that $c \overset{\beta}{\to} —— x_2$ fails. Since $c \overset{\alpha}{\to} —— b, d$, we must have $S_\delta(u) \cap S_\alpha(a) \cap S_\beta(c) \subseteq \{x_0, x_1\}$. By Lemma 6(3), we see that $\|S_\delta(u) \cap S_\alpha(a) \cap S_\beta(c)\| = 2$, so that $c \overset{\beta}{\to} —— x_0, x_1$. But this gives a monochromatic triangle, since $x_0 \overset{\beta}{\to} —— x_1$. Thus, we have

$$c \overset{\beta}{\to} —— x_0, x_2, y_0, y_2,$$

as desired.

Now, we note the following:

$$\text{Either } x_0 \overset{\gamma}{\to} —— y_2 \text{ or } x_0 \overset{\delta}{\to} —— y_2 \text{ since } x_0, y_2 \in S_\alpha(a) \cap S_\beta(c).$$

$$\text{Either } x_2 \overset{\gamma}{\to} —— y_0 \text{ or } x_2 \overset{\delta}{\to} —— y_0 \text{ since } x_2, y_0 \in S_\alpha(a) \cap S_\beta(c).$$

Note further that $b, x_0, x_1, x_2, d, y_2, y_1, y_0 \in S_\alpha(a)$. Note also that the following hold:

$$\{x_0, x_1, x_2\} \cap \{y_2, y_1, y_0\} = \emptyset$$
$$P_{\beta,\gamma}(b, x_0, x_1, x_2, d)$$
$$P_{\beta,\gamma}(d, y_2, y_1, y_0, b)$$

Thus, we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (b, x_0, x_1, x_2, d, y_2, y_1, y_0)$$

to see that $\|S_\alpha(a)\| \leq 14$.

But this is easily seen to be impossible. If $\|S_\delta(u) \cap S_\delta(v)\| = 4$, then $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$, so that $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(a)\| = 1$ so that, by Proposition 3, we have $\|S_\beta(a)\| \leq 14$, which is impossible, by Proposition 4, since $\|S_\alpha(a)\| \leq 14$. If, on the other hand, we have $\|S_\delta(u) \cap S_\delta(v)\| = 6$, then we have, by Proposition 17, that $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(a) \in \{1, 3\}$ for some $\eta \in \{\beta, \gamma\}$, since $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(a)\| = 2$. Thus, we have $\|S_\eta(a)\| \leq 14$ for some $\eta \in \{\beta, \gamma\}$, which is impossible, by Proposition 4, since $\|S_\alpha(a)\| \leq 14$.

The proof is complete. □

Proposition 19. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose that $a, b, c, d \in S_\delta(u) \cap S_\delta(v)$. If $a \stackrel{\gamma}{\longrightarrow} b \stackrel{\beta}{\longrightarrow} c \stackrel{\beta}{\longrightarrow} d \stackrel{\beta}{\longrightarrow} a$ and $a \stackrel{\alpha}{\longrightarrow} c$ and $b \stackrel{\alpha}{\longrightarrow} d$, then $\|S_\delta(u) \cap S_\delta(v)\| \neq 6$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 6$. Since $a \stackrel{\beta}{\longrightarrow} d \stackrel{\beta}{\longrightarrow} c \stackrel{\beta}{\longrightarrow} b$, we have, by Proposition 17,

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| \in \{2, 3\}.$$

First, we show that either $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = 2$ or $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| = 2$. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| = 3$. Thus, there exists $x, y \in S_\delta(u) \cap S_\delta(v) \sim \{a, b, c, d\}$ with $S_\delta(u) \cap S_\delta(v) \cap S_\beta(c) = \{b, d, x\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(d) = \{a, c, y\}$. Since $x \stackrel{\beta}{\longrightarrow} c \stackrel{\beta}{\longrightarrow} d \stackrel{\beta}{\longrightarrow} y$, we must have $x \neq y$. Thus, we have $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, x, y\}$. Since $c \stackrel{\beta}{\longrightarrow} x, b, d$ and $c \stackrel{\alpha}{\longrightarrow} a$, we see, by Proposition 17, that $c \stackrel{\alpha}{\longrightarrow} y$. Since $d \stackrel{\beta}{\longrightarrow} y, a, c$ and $d \stackrel{\alpha}{\longrightarrow} b$, we see, by Proposition 17, that $d \stackrel{\alpha}{\longrightarrow} x$. Note that $x \stackrel{\gamma}{\longrightarrow} b$ since $x, b \in S_\delta(u) \cap S_\alpha(d) \cap S_\beta(c)$, and that $y \stackrel{\gamma}{\longrightarrow} a$ since $y, a \in S_\delta(u) \cap S_\alpha(c) \cap S_\beta(d)$. We cannot have $b \stackrel{\beta}{\longrightarrow} y$, since, if we did, we would get a contradiction with Proposition 18, by consideration of $y \stackrel{\beta}{\longrightarrow} d \stackrel{\beta}{\longrightarrow} c \stackrel{\beta}{\longrightarrow} b \stackrel{\beta}{\longrightarrow} y$ and $d \stackrel{\alpha}{\longrightarrow} b$ and $y \stackrel{\alpha}{\longrightarrow} c$. Thus, we have $b \stackrel{\alpha}{\longrightarrow} y$, since $b, y \in S_\delta(u) \cap S_\gamma(a)$. Also, we cannot have $x \stackrel{\beta}{\longrightarrow} a$, since, if we did, we would get a contradiction with Proposition 18, by consideration of $x \stackrel{\beta}{\longrightarrow} c \stackrel{\beta}{\longrightarrow} d \stackrel{\beta}{\longrightarrow} a \stackrel{\beta}{\longrightarrow} x$ and $c \stackrel{\alpha}{\longrightarrow} a$ and $x \stackrel{\alpha}{\longrightarrow} d$. Thus, we have $a \stackrel{\alpha}{\longrightarrow} x$, since $a, x \in S_\delta(u) \cap S_\gamma(b)$. Now, we consider the color of the edge from x to y. If $x \stackrel{\alpha}{\longrightarrow} y$, we get a contradiction with Proposition 17, since then we would have $x \stackrel{\alpha}{\longrightarrow} a, d, y$ and $x \stackrel{\beta}{\longrightarrow} c$ and $x \stackrel{\gamma}{\longrightarrow} b$. If $x \stackrel{\beta}{\longrightarrow} y$, we get a contradiction with Proposition 18, by consideration of $x \stackrel{\beta}{\longrightarrow} y \stackrel{\beta}{\longrightarrow} d \stackrel{\beta}{\longrightarrow} c \stackrel{\beta}{\longrightarrow} x$ and $x \stackrel{\alpha}{\longrightarrow} d$ and $y \stackrel{\alpha}{\longrightarrow} c$. If $x \stackrel{\gamma}{\longrightarrow} y$, we get a contradiction with Proposition 18, by consideration of $x \stackrel{\gamma}{\longrightarrow} y \stackrel{\gamma}{\longrightarrow} a \stackrel{\gamma}{\longrightarrow} b \stackrel{\gamma}{\longrightarrow} x$ and $x \stackrel{\alpha}{\longrightarrow} a$ and $y \stackrel{\alpha}{\longrightarrow} b$. Thus, we must have $x \stackrel{\delta}{\longrightarrow} y$. But this gives a monochromatic triangle, since $x \stackrel{\delta}{\longrightarrow} u \stackrel{\delta}{\longrightarrow} y$. thus, we have shown that either $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = 2$ or $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| = 2$.

Without loss of generality, we may suppose that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = 2.$$

By Lemma 4, there exist $x_0, x_1, x_2 \in S_\delta(u) \cap S_\beta(c)$ and $y_0, y_1, y_2 \in S_\delta(v) \cap S_\beta(c)$ such that both $P_{\alpha,\gamma}(b, x_0, x_1, x_2, d)$ and $P_{\alpha,\gamma}(b, y_0, y_1, y_2, d)$ hold.

Now, we show that $a \stackrel{\alpha}{\longrightarrow} x_0, x_2, y_0, y_2$. Suppose not. Without loss of generality, we may suppose that either $a \stackrel{\alpha}{\longrightarrow} x_0$ fails or else $a \stackrel{\alpha}{\longrightarrow} x_2$ fails.

Suppose that $a \stackrel{\alpha}{\longrightarrow} x_0$ fails. Since $a \stackrel{\gamma}{\longrightarrow} b$ and $a \stackrel{\beta}{\longrightarrow} d$, we must have $S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a) \subseteq \{x_1, x_2\}$. By Lemma 6(3), we see that $\|S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a)\| = 2$, so that $a \stackrel{\alpha}{\longrightarrow} x_1, x_2$. But this gives a monochromatic triangle, since $x_1 \stackrel{\alpha}{\longrightarrow} x_2$.

Suppose that $a \stackrel{\alpha}{\longrightarrow} x_2$ fails. Since $a \stackrel{\gamma}{\longrightarrow} b$ and $a \stackrel{\beta}{\longrightarrow} d$, we must have $S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a) \subseteq \{x_0, x_1\}$. By Lemma 6(3), we see that $\|S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a)\| = 2$, so that $a \stackrel{\alpha}{\longrightarrow} x_0, x_1$. But this gives a monochromatic triangle, since $x_0 \stackrel{\alpha}{\longrightarrow} x_1$.

Thus, we have

$$a \stackrel{\alpha}{\longrightarrow} x_0, x_2, y_0, y_2,$$

as desired.

Now, we note the following:

$$\text{Either } x_0 \xrightarrow{\gamma} y_2 \text{ or } x_0 \xrightarrow{\delta} y_2 \text{ since } x_0, y_2 \in S_\beta(c) \cap S_\alpha(a).$$

$$\text{Either } x_2 \xrightarrow{\gamma} y_0 \text{ or } x_2 \xrightarrow{\delta} y_0 \text{ since } x_2, y_0 \in S_\beta(c) \cap S_\alpha(a).$$

Note further that $b, x_0, x_1, x_2, d, y_2, y_1, y_0 \in S_\beta(c)$. Note also that the following hold:

$$\{x_0, x_1, x_2\} \cap \{y_2, y_1, y_0\} = \emptyset$$
$$P_{\alpha,\gamma}(b, x_0, x_1, x_2, d)$$
$$P_{\alpha,\gamma}(d, y_2, y_1, y_0, b)$$

Thus, we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (b, x_0, x_1, x_2, d, y_2, y_1, y_0)$$

to see that $\|S_\beta(c)\| \leq 14$.

But this is easily seen to be impossible. By Proposition 17, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(c) \in \{1, 3\}$ for some $\eta \in \{\alpha, \gamma\}$, since $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c))\| = 2$. Thus, we have $\|S_\eta(c)\| \leq 14$ for some $\eta \in \{\alpha, \gamma\}$, which is impossible, by Proposition 4, since $\|S_\beta(c)\| \leq 14$.

The proof is complete. □

Proposition 20. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 6$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$(45) \quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [2, 2, 1].$$

Proof. Suppose not. Then, by Proposition 17, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 3,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 2,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

The induced coloring on the complete graph with vertex set $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)$ has three edges, each of which is colored with either β or γ. Thus, there exists some $y \in S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)$ such that both edges from y are the same color.

By Lemma 5, there exist $x_0, \ldots, x_{15} \in S_\delta(u)$ such that one of the following cases hold:

1. $C^0_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_1, x_5\}$.
2. $C^0_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_2, x_4\}$.
3. $C^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_1, x_5\}$.
4. $C^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_2, x_4\}$.
5. $C^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_2$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_2, x_4, x_5\}$.
6. $C^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_2$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_2, x_1, x_3\}$.
7. $C^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_1$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_1, x_3, x_4\}$.
8. $C^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ where $x = x_0$ and $y = x_1$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_1, x_2, x_5\}$.

Note that in all cases, we have

$$S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \subseteq \{x_6, x_7, x_8, x_9, x_{10}\}.$$

For case (1), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_1 \stackrel{\gamma}{\to}\text{---}\, x_5$ and $x_3 \stackrel{\gamma}{\to}\text{---}\, x_8$ and $x_1 \stackrel{\beta}{\to}\text{---}\, x_8 \stackrel{\beta}{\to}\text{---}\, x_5 \stackrel{\beta}{\to}\text{---}\, x_3 \stackrel{\beta}{\to}\text{---}\, x_1$. Also, $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \stackrel{\beta}{\to}\text{---}\, x_7$ and $x_1 \stackrel{\beta}{\to}\text{---}\, x_3$ and $x_0 \stackrel{\alpha}{\to}\text{---}\, x_1 \stackrel{\alpha}{\to}\text{---}\, x_7 \stackrel{\alpha}{\to}\text{---}\, x_3 \stackrel{\alpha}{\to}\text{---}\, x_0$. Also, $x_9 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \stackrel{\alpha}{\to}\text{---}\, x_5 \stackrel{\alpha}{\to}\text{---}\, x_9 \stackrel{\alpha}{\to}\text{---}\, x_3 \stackrel{\alpha}{\to}\text{---}\, x_0$ and $x_0 \stackrel{\beta}{\to}\text{---}\, x_9$ and $x_5 \stackrel{\beta}{\to}\text{---}\, x_3$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Proposition 17, by consideration of $x_3 \stackrel{\beta}{\to}\text{---}\, x_1, x_5, x_6, x_{10}$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (1) lead to contradiction.

For case (2), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \stackrel{\alpha}{\to}\text{---}\, x_2 \stackrel{\alpha}{\to}\text{---}\, x_8 \stackrel{\alpha}{\to}\text{---}\, x_4 \stackrel{\alpha}{\to}\text{---}\, x_0$ and $x_0 \stackrel{\beta}{\to}\text{---}\, x_8$ and $x_2 \stackrel{\beta}{\to}\text{---}\, x_4$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Lemma 8, by consideration of $x_0 \stackrel{\alpha}{\to}\text{---}\, x_3$ and $x_6 \stackrel{\alpha}{\to}\text{---}\, x_{10}$ and $x_0 \stackrel{\beta}{\to}\text{---}\, x_{10} \stackrel{\beta}{\to}\text{---}\, x_3 \stackrel{\beta}{\to}\text{---}\, x_6 \stackrel{\beta}{\to}\text{---}\, x_0$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_7\}$, by Lemma 7, by consideration of $x_4 \stackrel{\beta}{\to}\text{---}\, x_2, x_6, x_7$ and $x_4 \stackrel{\alpha}{\to}\text{---}\, x_0$ and $x_4 \stackrel{\gamma}{\to}\text{---}\, x_3$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Lemma 7, by consideration of $x_2 \stackrel{\beta}{\to}\text{---}\, x_4, x_9, x_{10}$ and $x_2 \stackrel{\alpha}{\to}\text{---}\, x_0$ and $x_2 \stackrel{\gamma}{\to}\text{---}\, x_3$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Lemma 7, by consideration of $x_6 \stackrel{\beta}{\to}\text{---}\, x_0, x_3, x_4$ and $x_6 \stackrel{\alpha}{\to}\text{---}\, x_2$ and $x_6 \stackrel{\gamma}{\to}\text{---}\, x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_7, x_{10}\}$, by Lemma 7, by consideration of $x_{10} \stackrel{\beta}{\to}\text{---}\, x_0, x_2, x_3$ and $x_{10} \stackrel{\alpha}{\to}\text{---}\, x_4$ and $x_{10} \stackrel{\gamma}{\to}\text{---}\, x_7$. Thus, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{x_7, x_9\}$ for case (.) We handle this case below.

For case (3), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_1 \stackrel{\beta}{\to}\text{---}\, x_8 \stackrel{\beta}{\to}\text{---}\, x_5 \stackrel{\beta}{\to}\text{---}\, x_3 \stackrel{\beta}{\to}\text{---}\, x_1$ and $x_1 \stackrel{\gamma}{\to}\text{---}\, x_5$ and $x_3 \stackrel{\gamma}{\to}\text{---}\, x_8$. Also, $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \stackrel{\beta}{\to}\text{---}\, x_7$ and $x_1 \stackrel{\beta}{\to}\text{---}\, x_3$ and $x_0 \stackrel{\alpha}{\to}\text{---}\, x_1 \stackrel{\alpha}{\to}\text{---}\, x_7 \stackrel{\alpha}{\to}\text{---}\, x_3 \stackrel{\alpha}{\to}\text{---}\, x_0$. Also, $x_9 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \stackrel{\alpha}{\to}\text{---}\, x_5 \stackrel{\alpha}{\to}\text{---}\, x_9 \stackrel{\alpha}{\to}\text{---}\, x_3 \stackrel{\alpha}{\to}\text{---}\, x_0$ and $x_0 \stackrel{\beta}{\to}\text{---}\, x_9$ and $x_5 \stackrel{\beta}{\to}\text{---}\, x_3$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Proposition 17, by consideration of $x_3 \stackrel{\beta}{\to}\text{---}\, x_1, x_5, x_6, x_{10}$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (3) lead to contradiction.

For case (4), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \stackrel{\alpha}{\to}\text{---}\, x_2 \stackrel{\alpha}{\to}\text{---}\, x_8 \stackrel{\alpha}{\to}\text{---}\, x_4 \stackrel{\alpha}{\to}\text{---}\, x_0$ and $x_0 \stackrel{\beta}{\to}\text{---}\, x_8$ and $x_2 \stackrel{\beta}{\to}\text{---}\, x_4$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Lemma 8, by consideration of $x_0 \stackrel{\alpha}{\to}\text{---}\, x_3$ and $x_6 \stackrel{\alpha}{\to}\text{---}\, x_{10}$ and $x_0 \stackrel{\beta}{\to}\text{---}\, x_{10} \stackrel{\beta}{\to}\text{---}\, x_3 \stackrel{\beta}{\to}\text{---}\, x_6 \stackrel{\beta}{\to}\text{---}\, x_0$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_7\}$, by Lemma 9, by consideration of $x_0 \stackrel{\alpha}{\to}\text{---}\, x_2$ and $x_6 \stackrel{\alpha}{\to}\text{---}\, x_7$ and $x_7 \stackrel{\gamma}{\to}\text{---}\, x_2 \stackrel{\beta}{\to}\text{---}\, x_6 \stackrel{\beta}{\to}\text{---}\, x_0 \stackrel{\beta}{\to}\text{---}\, x_7$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Lemma 9, by consideration of $x_9 \stackrel{\gamma}{\to}\text{---}\, x_4 \stackrel{\beta}{\to}\text{---}\, x_{10} \stackrel{\beta}{\to}\text{---}\, x_0 \stackrel{\beta}{\to}\text{---}\, x_9$ and $x_0 \stackrel{\alpha}{\to}\text{---}\, x_4$ and $x_{10} \stackrel{\alpha}{\to}\text{---}\, x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Lemma 7, by consideration of $x_6 \stackrel{\beta}{\to}\text{---}\, x_0, x_3, x_4$ and $x_6 \stackrel{\alpha}{\to}\text{---}\, x_2$ and $x_6 \stackrel{\gamma}{\to}\text{---}\, x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_7, x_{10}\}$, by Lemma 7, by consideration of $x_{10} \stackrel{\alpha}{\to}\text{---}\, x_4$ and $x_{10} \stackrel{\gamma}{\to}\text{---}\, x_7$ and $x_{10} \stackrel{\beta}{\to}\text{---}\, x_0, x_2, x_3$. Thus, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{x_7, x_9\}$, for case (4). We handle this case below.

For case (5), first note that $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_2 \overset{\beta}{\to}\text{—} x_5 \overset{\beta}{\to}\text{—} x_7 \overset{\beta}{\to}\text{—} x_4 \overset{\beta}{\to}\text{—} x_2$ and $x_2 \overset{\gamma}{\to}\text{—} x_7$ and $x_4 \overset{\gamma}{\to}\text{—} x_5$. Also, $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \overset{\beta}{\to}\text{—} x_8$ and $x_2 \overset{\beta}{\to}\text{—} x_4$ and $x_0 \overset{\alpha}{\to}\text{—} x_2 \overset{\alpha}{\to}\text{—} x_8 \overset{\alpha}{\to}\text{—} x_4 \overset{\alpha}{\to}\text{—} x_0$. Also, $x_{10} \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 19, by consideration of $x_{10} \overset{\gamma}{\to}\text{—} x_5 \overset{\alpha}{\to}\text{—} x_0 \overset{\alpha}{\to}\text{—} x_2 \overset{\alpha}{\to}\text{—} x_{10}$ and $x_0 \overset{\beta}{\to}\text{—} x_{10}$ and $x_2 \overset{\beta}{\to}\text{—} x_5$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Proposition 17, by consideration of $x_2 \overset{\beta}{\to}\text{—} x_4, x_5, x_6, x_9$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (5) lead to contradiction.

For case (6), first note that $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \overset{\alpha}{\to}\text{—} x_1 \overset{\alpha}{\to}\text{—} x_7 \overset{\alpha}{\to}\text{—} x_3 \overset{\alpha}{\to}\text{—} x_0$ and $x_0 \overset{\beta}{\to}\text{—} x_7$ and $x_1 \overset{\beta}{\to}\text{—} x_3$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_8\}$, by Proposition 19, by consideration of $x_2 \overset{\alpha}{\to}\text{—} x_8 \overset{\gamma}{\to}\text{—} x_6 \overset{\gamma}{\to}\text{—} x_1 \overset{\gamma}{\to}\text{—} x_2$ and $x_1 \overset{\beta}{\to}\text{—} x_8$ and $x_2 \overset{\beta}{\to}\text{—} x_6$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Proposition 17, by consideration of $x_9 \overset{\beta}{\to}\text{—} x_0, x_1, x_2$ and $x_9 \overset{\alpha}{\to}\text{—} x_3$ and $x_9 \overset{\gamma}{\to}\text{—} x_6$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Proposition 17, by consideration of $x_3 \overset{\beta}{\to}\text{—} x_1, x_6, x_{10}$ and $x_3 \overset{\alpha}{\to}\text{—} x_0$ and $x_3 \overset{\gamma}{\to}\text{—} x_2$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_9\}$, by Proposition 17, by consideration of $x_1 \overset{\beta}{\to}\text{—} x_3, x_8, x_9$ and $x_1 \overset{\alpha}{\to}\text{—} x_0$ and $x_1 \overset{\gamma}{\to}\text{—} x_2$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_{10}\}$, by Proposition 19, by consideration of $x_1 \overset{\beta}{\to}\text{—} x_8 \overset{\alpha}{\to}\text{—} x_2 \overset{\alpha}{\to}\text{—} x_{10} \overset{\alpha}{\to}\text{—} x_1$ and $x_1 \overset{\gamma}{\to}\text{—} x_2$ and $x_8 \overset{\gamma}{\to}\text{—} x_{10}$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Proposition 19, by consideration of $x_2 \overset{\gamma}{\to}\text{—} x_3 \overset{\alpha}{\to}\text{—} x_9 \overset{\alpha}{\to}\text{—} x_{10} \overset{\alpha}{\to}\text{—} x_2$ and $x_2 \overset{\beta}{\to}\text{—} x_9$ and $x_3 \overset{\beta}{\to}\text{—} x_{10}$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (6) lead to contradiction.

For case (7), first note that $x_6 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 19, by consideration of $x_6 \overset{\alpha}{\to}\text{—} x_4 \overset{\beta}{\to}\text{—} x_1 \overset{\beta}{\to}\text{—} x_3 \overset{\beta}{\to}\text{—} x_6$ and $x_1 \overset{\gamma}{\to}\text{—} x_6$ and $x_3 \overset{\gamma}{\to}\text{—} x_4$. Also, $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \overset{\beta}{\to}\text{—} x_7$ and $x_1 \overset{\beta}{\to}\text{—} x_3$ and $x_0 \overset{\alpha}{\to}\text{—} x_1 \overset{\alpha}{\to}\text{—} x_7 \overset{\alpha}{\to}\text{—} x_3 \overset{\alpha}{\to}\text{—} x_0$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_9\}$, by Proposition 17, by consideration of $x_1 \overset{\beta}{\to}\text{—} x_3, x_4, x_8, x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Proposition 19, by consideration of $x_0 \overset{\alpha}{\to}\text{—} x_4$ and $x_9 \overset{\alpha}{\to}\text{—} x_{10}$ and $x_4 \overset{\gamma}{\to}\text{—} x_9 \overset{\beta}{\to}\text{—} x_0 \overset{\beta}{\to}\text{—} x_{10} \overset{\beta}{\to}\text{—} x_4$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_{10}\}$, by Proposition 17, by consideration of $x_{10} \overset{\beta}{\to}\text{—} x_0, x_3, x_4$ and $x_{10} \overset{\alpha}{\to}\text{—} x_1$ and $x_{10} \overset{\gamma}{\to}\text{—} x_8$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (7) lead to contradiction.

For case (8), first note that $x_{10} \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 19, by consideration of $x_5 \overset{\gamma}{\to}\text{—} x_{10} \overset{\alpha}{\to}\text{—} x_2 \overset{\alpha}{\to}\text{—} x_0 \overset{\alpha}{\to}\text{—} x_5$ and $x_0 \overset{\beta}{\to}\text{—} x_{10}$ and $x_5 \overset{\beta}{\to}\text{—} x_2$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_7\}$, by Proposition 19, by consideration of $x_7 \overset{\gamma}{\to}\text{—} x_2 \overset{\beta}{\to}\text{—} x_6 \overset{\beta}{\to}\text{—} x_0 \overset{\beta}{\to}\text{—} x_7$ and $x_0 \overset{\alpha}{\to}\text{—} x_2$ and $x_6 \overset{\alpha}{\to}\text{—} x_7$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_8\}$, by Proposition 17, by consideration of $x_1 \overset{\gamma}{\to}\text{—} x_2, x_5, x_6$ and $x_1 \overset{\alpha}{\to}\text{—} x_0$ and $x_1 \overset{\beta}{\to}\text{—} x_8$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Proposition 17, by consideration of $x_1 \overset{\gamma}{\to}\text{—} x_2, x_5, x_6$ and $x_1 \overset{\alpha}{\to}\text{—} x_0$ and $x_1 \overset{\beta}{\to}\text{—} x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_7, x_8\}$, by Proposition 17, by consideration of $x_5 \overset{\alpha}{\to}\text{—} x_0$ and $x_5 \overset{\gamma}{\to}\text{—} x_1$ and $x_5 \overset{\beta}{\to}\text{—} x_2, x_7, x_8$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_7, x_9\}$, by Proposition 17, by consideration of $x_9 \overset{\beta}{\to}\text{—} x_0, x_1, x_2$ and $x_9 \overset{\alpha}{\to}\text{—} x_5$ and $x_9 \overset{\gamma}{\to}\text{—} x_7$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_9\}$, by Proposition 18, by consideration of $x_0 \overset{\beta}{\to}\text{—} x_8 \overset{\beta}{\to}\text{—} x_1 \overset{\beta}{\to}\text{—} x_9 \overset{\beta}{\to}\text{—} x_0$ and $x_0 \overset{\alpha}{\to}\text{—} x_1$ and $x_8 \overset{\alpha}{\to}\text{—} x_9$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (8) lead to contradiction.

We have thus shown that for some $i \in \{0,1\}$, there exist $x_0, \ldots, x_{15} \in S_\delta(u)$ with $C^i_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ such that $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) = \{x_0, x_2, x_3, x_4, x_7, x_9\}$.

Switching the roles of u and v in this argument shows that for some $j \in \{0,1\}$, there exist $y_0, \ldots, y_{15} \in S_\delta(v)$ with $C^j_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $x = y_0$ and $y = y_3$ and $S_\delta(u) \cap S_\delta(v) = \{y_0, y_2, y_3, y_4, y_7, y_9\}$.

Since $x = x_0 = y_0$, there are two possibilities. The first is

$$\begin{aligned} x_0 &= y_0 \\ x_2 &= y_2 \\ x_3 &= y_3 \\ x_4 &= y_4 \\ x_7 &= y_7 \\ x_9 &= y_9 \end{aligned}$$

The second is

$$\begin{aligned} x_0 &= y_0 \\ x_2 &= y_4 \\ x_3 &= y_3 \\ x_4 &= y_2 \\ x_7 &= y_9 \\ x_9 &= y_7 \end{aligned}$$

Actually, these possibilities represent isomorphic situations. Thus, we may assume, without loss of generality, that the former possibility holds. As luck (and a clever choice of notation!) would have it, we may ignore the indices i and j. We will use only edges where the twistedness or untwistedness does not matter. Let $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e, f\}$, where the following hold:

$$\begin{aligned} a &= x_0 = y_0 \\ b &= x_2 = y_2 \\ c &= x_3 = y_3 \\ d &= x_4 = y_4 \\ e &= x_7 = y_7 \\ f &= x_9 = y_9 \end{aligned}$$

Now, we note the following:

$$x_{11} \xrightarrow{\delta} y_{11} \text{ since } x_{11}, y_{11} \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(e).$$
$$x_{12} \xrightarrow{\delta} y_{12} \text{ since } x_{12}, y_{12} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$
$$x_{14} \xrightarrow{\delta} y_{14} \text{ since } x_{14}, y_{14} \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(d).$$
$$x_{15} \xrightarrow{\delta} y_{15} \text{ since } x_{15}, y_{15} \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(f).$$
$$\text{Either } x_{11} \xrightarrow{\beta} y_{12} \text{ or } x_{11} \xrightarrow{\delta} y_{12} \text{ since } x_{11}, y_{12} \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_{12} \xrightarrow{\beta} y_{11} \text{ or } x_{12} \xrightarrow{\delta} y_{11} \text{ since } x_{12}, y_{11} \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_{14} \xrightarrow{\beta} y_{15} \text{ or } x_{14} \xrightarrow{\delta} y_{15} \text{ since } x_{14}, y_{15} \in S_\gamma(a) \cap S_\alpha(b).$$
$$\text{Either } x_{15} \xrightarrow{\beta} y_{14} \text{ or } x_{15} \xrightarrow{\delta} y_{14} \text{ since } x_{15}, y_{14} \in S_\gamma(a) \cap S_\alpha(b).$$
$$\text{Either } x_{11} \xrightarrow{\beta} y_{15} \text{ or } x_{11} \xrightarrow{\delta} y_{15} \text{ since } x_{11}, y_{15} \in S_\gamma(a) \cap S_\alpha(c).$$
$$\text{Either } x_{15} \xrightarrow{\beta} y_{11} \text{ or } x_{15} \xrightarrow{\delta} y_{11} \text{ since } x_{15}, y_{11} \in S_\gamma(a) \cap S_\alpha(c).$$

Note further that $x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\} \cap \{y_{11}, y_{12}, y_{13}, y_{14}, y_{15}\} = \emptyset$$
$$P_{\beta,\alpha}(x_{11}, x_{12}, x_{13}, x_{14}, x_{15})$$
$$P_{\beta,\alpha}(y_{11}, y_{12}, y_{13}, y_{14}, y_{15})$$

Since $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(a)\| = 3$, we see, by Proposition 3, that $\|S_\alpha(a)\| \leq 14$, so that, by Proposition 1, we have

$$\|S_\gamma(a)\| \geq 15.$$

Thus, we may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{11}, x_{12}, x_{13}, x_{14}, x_{15})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{11}, y_{12}, y_{13}, y_{14}, y_{15})$$

to see that

$$\text{either } M^0_{\beta,\alpha,\delta}(x_{12}, x_{11}, x_{15}, x_{14}, x_{13}, y_{12}, y_{11}, y_{15}, y_{14}, y_{13})$$
$$\text{or } M^0_{\beta,\alpha,\delta}(x_{13}, x_{12}, x_{11}, x_{15}, x_{14}, y_{13}, y_{12}, y_{11}, y_{15}, y_{14})$$
$$\text{or } M^2_{\beta,\alpha,\delta}(x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15})$$
$$\text{or } M^2_{\beta,\alpha,\delta}(x_{15}, x_{14}, x_{13}, x_{12}, x_{11}, y_{15}, y_{14}, y_{13}, y_{12}, y_{11}).$$

Thus, we see that

$$x_{12} \xrightarrow{\alpha} y_{14} \text{ and } x_{14} \xrightarrow{\alpha} y_{12}.$$

Finally, note that $d, y_{14}, y_8, y_{12}, b, x_{12}, x_8, x_{14} \in S_\gamma(c)$. Note also that the following hold:

$$\{y_{14}, y_8, y_{12}\} \cap \{x_{12}, x_8, x_{14}\} = \emptyset$$
$$P_{\beta,\alpha}(d, y_{14}, y_8, y_{12}, b)$$
$$P_{\beta,\alpha}(b, x_{12}, x_8, x_{14}, d)$$

Since $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(c)\| = 3$, we see, by Proposition 3, that $\|S_\alpha(c)\| \leq 14$, so that, by Proposition 1, we have

$$\|S_\gamma(c)\| \geq 15.$$

Thus, we may now apply Lemma 12 with

$$(u_1, \ldots, u_8) = (d, y_{14}, y_8, y_{12}, b, x_{12}, x_8, x_{14})$$

to produce the desired contradiction. The proof is complete. □

Theorem 10. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 6$.*

Proof. For any $\eta \in \{\alpha, \beta, \gamma\}$, we define

$$U_\eta = \left\{ x \in S_\delta(u) \cap S_\delta(v) \;\middle|\; \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 1 \right\}.$$

By Proposition 20, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \uplus U_\beta \uplus U_\gamma.$$

Thus, we have

$$6 = \|U_\alpha\| + \|U_\beta\| + \|U_\gamma\|.$$

Note that the number of η colored edges in the induced coloring on the complete graph with vertex set $S_\delta(u) \cap S_\delta(v)$ is

$$\frac{\|U_\eta\| + 2 \cdot (6 - \|U_\eta\|)}{2}.$$

Thus, $\|U_\eta\|$ is even for each $\eta \in \{\alpha, \beta, \gamma\}$ It follows that

$$(\|U_\alpha\|, \|U_\beta\|, \|U_\gamma\|) \in [6, 0, 0] \cup [4, 2, 0] \cup [2, 2, 2].$$

Next, we show that $\|U_\eta\| \neq 2$ for all $\eta \in \{\alpha, \beta, \gamma\}$. Suppose not. Then $\|U_\eta\| = 2$ for some $\eta \in \{\alpha, \beta, \gamma\}$. We consider the graph whose edges are the η colored edges of $S_\delta(u) \cap S_\delta(v)$. This graph is a triangle free graph with 2 vertices of degree 1 and 4 vertices of degree 2. Such a graph has exactly 5 edges. There is only one such graph up to isomorphism. Thus, we may suppose that $S_\delta(u) \cap S_\delta(v) = \{w_0, w_1, w_2, w_3, x, y\}$ with $w_0 \xrightarrow{\eta} w_1 \xrightarrow{\eta} w_2 \xrightarrow{\eta} w_3 \xrightarrow{\eta} w_0 \xrightarrow{\eta}$ and $x \xrightarrow{\eta} y$. Lemma 8 then gives a contradiction. Thus, we have $\|U_\eta\| \neq 2$ for all $\eta \in \{\alpha, \beta, \gamma\}$.

It follows that

$$(\|U_\alpha\|, \|U_\beta\|, \|U_\gamma\|) \in [6, 0, 0],$$

so that, we may assume, without loss of generality, that

$$\|U_\alpha\| = 6,$$

and

$$\|U_\beta\| = \|U_\gamma\| = 0.$$

Such a coloring on a complete graph with 6 vertices is clearly impossible.

The proof is complete. □

4.11. **3.11 Attaching Sets of Cardinality** 5.

Proposition 21. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 5$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [4, 0, 0] \cup [2, 2, 0]. \tag{46}$$

Proof. First note that for any $\eta \in \{\alpha,\beta,\gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \le R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$(47)\quad S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big),$$

so that

$$(48)\quad 5 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|.$$

Thus, we have

$$(49)\quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [4,0,0] \cup [3,1,1] \cup [2,2,0] \cup [2,1,1].$$

The proposition now follows by an application of Proposition 4. ☐

Proposition 22. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 5$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$(50)\quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [2,2,0].$$

Proof. Suppose not. Then by Proposition 21, there exist some $x \in S_\delta(u) \cap S_\delta(v)$ such that $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 4$ for some $\eta \in \{\alpha,\beta,\gamma\}$. We may suppose that $S_\delta(u) \cap S_\delta(v) = \{x, w_0, w_1, w_2, w_3\}$. Clearly, we cannot have $w_0 \overset{\eta}{\to}$ —— w_i for any $i \in \{1,2,3\}$. Thus, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_0)\| = 1$, which contradicts Proposition 21.

The proof is complete. ☐

Proposition 23. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{12} = y_{12}$, $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, $d = x_{15} = y_{14}$, and $e = x_{11} = y_{13}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$\begin{gathered} u \longmapsto v \longmapsto u \\ a \longmapsto a \\ b \longmapsto e \longmapsto b \\ c \longmapsto d \longmapsto c \\ x_i \longmapsto y_i \longmapsto x_i \quad \text{for all} \quad i \in \{0, \ldots, 15\} \end{gathered}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}\alpha &\longmapsto \alpha\\ \beta &\longmapsto \beta\\ \gamma &\longmapsto \gamma\\ \delta &\longmapsto \delta\end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

First, we note the following:

$$\begin{aligned}
x_3 &\overset{\delta}{\to} \text{——}\ y_1 \text{ since } x_3, y_1 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(b).\\
x_{10} &\overset{\delta}{\to} \text{——}\ y_8 \text{ since } x_{10}, y_8 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(a).\\
x_3 &\overset{\delta}{\to} \text{——}\ y_7 \text{ since } x_3, y_7 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).\\
x_9 &\overset{\delta}{\to} \text{——}\ y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).\\
x_9 &\overset{\delta}{\to} \text{——}\ y_4 \text{ since } x_9, y_4 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).\\
\text{Either } x_3 &\overset{\beta}{\to} \text{——}\ y_8 \text{ or } x_3 \overset{\delta}{\to} \text{——}\ y_8 \text{ since } x_3, y_8 \in S_\gamma(c) \cap S_\alpha(e).\\
\text{Either } x_9 &\overset{\alpha}{\to} \text{——}\ y_8 \text{ or } x_9 \overset{\delta}{\to} \text{——}\ y_8 \text{ since } x_9, y_8 \in S_\gamma(c) \cap S_\beta(d).\\
\text{Either } x_9 &\overset{\alpha}{\to} \text{——}\ y_7 \text{ or } x_9 \overset{\delta}{\to} \text{——}\ y_7 \text{ since } x_9, y_7 \in S_\gamma(c) \cap S_\beta(b).
\end{aligned}$$

Note further that $x_3, x_1, x_9, x_0, x_{10}, y_1, y_4, y_7, y_0, y_8 \in S_\gamma(c)$. Note also that the following hold:

$$\begin{gathered}\{x_3, x_1, x_9, x_0, x_{10}\} \cap \{y_1, y_4, y_7, y_0, y_8\} = \emptyset\\ P_{\beta,\alpha}(x_3, x_1, x_9, x_0, x_{10})\\ P_{\beta,\alpha}(y_1, y_4, y_7, y_0, y_8)\end{gathered}$$

Now, we show that if $\|S_\gamma(c)\| \geq 15$, then $x_9 \overset{\delta}{\to}$ —— y_1, y_4, y_7, y_0, y_8. To that end, suppose that $\|S_\gamma(c)\| \geq 15$. Applying Lemma 14 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_4, y_7, y_0, y_8)$$

to see that either $x_9 \overset{\delta}{\to}$ —— y_1, y_4, y_7, y_0, y_8 or else there are exactly two δ-colored edges from x_9 to $\{y_1, y_4, y_7, y_0, y_8\}$. as the former is what we seek to show, we assume the latter. We already have two such edges above, namely $x_9 \overset{\delta}{\to}$ —— y_1 and $x_9 \overset{\delta}{\to}$ —— y_4. Thus, none of the edges from x_9 to $\{y_7, y_0, y_8\}$ can be δ-colored. But, by the above, we have either $x_9 \overset{\alpha}{\to}$ —— y_8 or $x_9 \overset{\delta}{\to}$ —— y_8. Also by the above, we have $x_9 \overset{\alpha}{\to}$ —— y_7 or $x_9 \overset{\delta}{\to}$ —— y_7. Thus, we must have $x_9 \overset{\alpha}{\to}$ —— y_8, y_7. But this is impossible, since otherwise the fact that $y_8 \overset{\alpha}{\to}$ —— y_7 would give a monochromatic triangle, thus giving a contradiction. Thus, we have shown that

$$\text{if } \|S_\gamma(c)\| \geq 15, \text{ then } x_9 \overset{\delta}{\to} \text{——}\ y_1, y_4, y_7, y_0, y_8, (1)$$

as desired.

Now, we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_4, y_7, y_0, y_8)$$

to see, with the help of (1), that $\|S_\gamma(c)\| \leq 14$ so that, by Proposition 1, we have

$$\|S_\alpha(c)\|, \|S_\beta(c)\| \geq 15. (2)$$

Next, we note the following:

$$x_4 \xrightarrow{\delta} y_9 \text{ since } x_4, y_9 \in S_\beta(c) \cap S_\alpha(a) \cap S_\gamma(b).$$

$$x_4 \xrightarrow{\delta} y_5 \text{ since } x_4, y_5 \in S_\beta(c) \cap S_\alpha(e) \cap S_\gamma(b).$$

Note further that $d, y_5, y_9, y_{10}, b, x_8, x_4, x_6 \in S_\beta(c)$. Note also that the following hold:

$$\{y_5, y_9, y_{10}\} \cap \{x_8, x_4, x_6\} = \emptyset$$

$$P_{\alpha,\gamma}(d, y_5, y_9, y_{10}, b)$$

$$P_{\alpha,\gamma}(b, x_8, x_4, x_6, d)$$

Since $\|S_\beta(c)\| \geq 15$ by (2), we may Apply Lemma 12 with

$$(u_1, \ldots, u_8) = (d, y_5, y_9, y_{10}, b, x_8, x_4, x_6)$$

to see that

$$N^2_{\delta,\gamma,\alpha}(d, x_6, x_4, x_8, b, y_{10}, y_9, y_5). (3)$$

Now, we note the following:

$$x_5 \xrightarrow{\delta} y_6 \text{ since } x_5, y_6 \in S_\alpha(c) \cap S_\beta(d) \cap S_\gamma(a).$$

$$x_7 \xrightarrow{\delta} y_6 \text{ since } x_7, y_6 \in S_\alpha(c) \cap S_\beta(b) \cap S_\gamma(a).$$

Note further that $a, y_2, y_6, y_3, e, x_7, x_5, x_2 \in S_\alpha(c)$. Note also that the following hold:

$$\{y_2, y_6, y_3\} \cap \{x_7, x_5, x_2\} = \emptyset$$

$$P_{\beta,\gamma}(a, y_2, y_6, y_3, e)$$

$$P_{\beta,\gamma}(e, x_7, x_5, x_2, a)$$

Since $\|S_\alpha(c)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_2, y_6, y_3, e, x_7, x_5, x_2)$$

to see that

$$N^2_{\delta,\gamma,\beta}(a, x_2, x_5, x_7, e, y_3, y_6, y_2). (4)$$

Now, we note the following:

$$x_6 \xrightarrow{\alpha} y_9 \text{ by (3).}$$

$$x_7 \xrightarrow{\delta} y_3 \text{ since } x_7, y_3 \in S_\beta(e) \cap S_\alpha(c) \cap S_\gamma(a).$$

$$x_1 \xrightarrow{\delta} y_3 \text{ since } x_1, y_3 \in S_\beta(e) \cap S_\alpha(b) \cap S_\gamma(d).$$

Note further that $a, y_9, y_3, y_7, d, x_6, x_7, x_1 \in S_\beta(e)$. Note also that the following hold:

$$\{y_9, y_3, y_7\} \cap \{x_6, x_7, x_1\} = \emptyset$$

$$P_{\alpha,\gamma}(a, y_9, y_3, y_7, d)$$

$$P_{\alpha,\gamma}(d, x_6, x_7, x_1, a)$$

We may now apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_9, y_3, y_7, d, x_6, x_7, x_1)$$

to see that $\|S_\beta(a)\| \leq 14$ so that, by Proposition 1, we have

$$\|S_\gamma(e)\| \geq 15.(5)$$

From (3), we see that

$$x_6 \overset{\alpha}{\to} \text{——} y_9.(6)$$

An application of the symmetry Θ to (6) gives

$$x_9 \overset{\alpha}{\to} \text{——} y_6.(7)$$

Finally, we note the following:

$$\begin{gathered}
x_5 \overset{\delta}{\to} \text{——} y_4 \text{ since } x_5, y_4 \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(d). \\
x_8 \overset{\delta}{\to} \text{——} y_{10} \text{ by (3).} \\
x_2 \overset{\delta}{\to} \text{——} y_2 \text{ by (4).} \\
x_5 \overset{\delta}{\to} \text{——} y_2 \text{ by (4).} \\
x_2 \overset{\beta}{\to} \text{——} y_6 \text{ by (4).} \\
x_9 \overset{\alpha}{\to} \text{——} y_6 \text{ by (7).}
\end{gathered}$$

Note further that $x_0, x_8, x_5, x_2, x_9, y_0, y_{10}, y_4, y_2, y_6 \in S_\gamma(e)$. Note also that the following hold:

$$\begin{gathered}
\{x_0, x_8, x_5, x_2, x_9\} \cap \{y_0, y_{10}, y_4, y_2, y_6\} = \emptyset \\
P_{\beta,\alpha}(x_0, x_8, x_5, x_2, x_9) \\
P_{\beta,\alpha}(y_0, y_{10}, y_4, y_2, y_6)
\end{gathered}$$

Since $\|S_\gamma(e)\| \geq 15$ by (5), we may apply Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_8, x_5, x_2, x_9)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_{10}, y_4, y_2, y_6)$$

to produce the desired contradiction. The proof is complete. $\square$

Proposition 24. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. If the graph has at least 62 vertices, then for any $w \in \{a, b, c, d, e\}$ we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted. (Note that both $\|S_\alpha(w)\|, \|S_\beta(w)\| \geq 15$ in this case, so that it makes sense to ask whether they are twisted in this last sentence.)*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto a$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_7 \longmapsto x_5 \longmapsto x_9 \longmapsto x_1$$
$$x_2 \longmapsto x_{10} \longmapsto x_4 \longmapsto x_6 \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1$$
$$y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

We assume that the graph has at least 62 vertices.

First, we note the following:

$$x_2 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(e).$$
$$x_6 \overset{\delta}{\to} \text{——} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$
$$x_{10} \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$
$$x_4 \overset{\delta}{\to} \text{——} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$
$$\text{Either } x_{10} \overset{\alpha}{\to} \text{——} y_2 \text{ or } x_{10} \overset{\delta}{\to} \text{——} y_2 \text{ since } x_{10}, y_2 \in S_\gamma(a) \cap S_\beta(e).$$
$$\text{Either } x_6 \overset{\alpha}{\to} \text{——} y_4 \text{ or } x_6 \overset{\delta}{\to} \text{——} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b).$$
$$\text{Either } x_2 \overset{\alpha}{\to} \text{——} y_{10} \text{ or } x_2 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_2, y_{10} \in S_\gamma(a) \cap S_\beta(e).$$
$$\text{Either } x_4 \overset{\alpha}{\to} \text{——} y_6 \text{ or } x_4 \overset{\delta}{\to} \text{——} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\beta(b).$$
$$\text{Either } x_2 \overset{\beta}{\to} \text{——} y_6 \text{ or } x_2 \overset{\delta}{\to} \text{——} y_6 \text{ since } x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$
$$\text{Either } x_4 \overset{\beta}{\to} \text{——} y_{10} \text{ or } x_4 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_4, y_{10} \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_6 \overset{\beta}{\to} \text{——} y_2 \text{ or } x_6 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$
$$\text{Either } x_{10} \overset{\beta}{\to} \text{——} y_4 \text{ or } x_{10} \overset{\delta}{\to} \text{——} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\alpha(d).$$

Note further that $x_2, x_6, x_0, x_{10}, x_4, y_2, y_6, y_0, y_{10}, y_4 \in S_\gamma(a)$. Note also that the following hold:

$$\{x_2, x_6, x_0, x_{10}, x_4\} \cap \{y_2, y_6, y_0, y_{10}, y_4\} = \emptyset$$
$$P_{\beta,\alpha}(x_2, x_6, x_0, x_{10}, x_4)$$
$$P_{\beta,\alpha}(y_2, y_6, y_0, y_{10}, y_4)$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_2, x_6, x_0, x_{10}, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_2, y_6, y_0, y_{10}, y_4)$$

to see that

$$\text{if } \|S_\gamma(a)\| \geq 15, \text{ then } M^0_{\beta,\alpha,\delta}(x_2, x_6, x_0, x_{10}, x_4, y_2, y_6, y_0, y_{10}, y_4).(1)$$

Next, we note the following:

$$x_1 \overset{\delta}{\to} \text{——} y_1 \text{ since } x_1, y_1 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(d).$$
$$x_9 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_9, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(a).$$
$$x_{10} \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c).$$
$$x_3 \overset{\delta}{\to} \text{——} y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(c) \cap S_\beta(a).$$
$$\text{Either } x_9 \overset{\alpha}{\to} \text{——} y_{10} \text{ or } x_9 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_9, y_{10} \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_{10} \overset{\alpha}{\to} \text{——} y_9 \text{ or } x_{10} \overset{\delta}{\to} \text{——} y_9 \text{ since } x_{10}, y_9 \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_9 \overset{\alpha}{\to} \text{——} y_3 \text{ or } x_9 \overset{\delta}{\to} \text{——} y_3 \text{ since } x_9, y_3 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_3 \overset{\alpha}{\to} \text{——} y_9 \text{ or } x_3 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_3, y_9 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_1 \overset{\beta}{\to} \text{——} y_9 \text{ or } x_1 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_1, y_9 \in S_\gamma(b) \cap S_\alpha(e).$$
$$\text{Either } x_9 \overset{\beta}{\to} \text{——} y_1 \text{ or } x_9 \overset{\delta}{\to} \text{——} y_1 \text{ since } x_9, y_1 \in S_\gamma(b) \cap S_\alpha(e).$$
$$\text{Either } x_{10} \overset{\beta}{\to} \text{——} y_3 \text{ or } x_{10} \overset{\delta}{\to} \text{——} y_3 \text{ since } x_{10}, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$
$$\text{Either } x_3 \overset{\beta}{\to} \text{——} y_{10} \text{ or } x_3 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d).$$

Note further that $x_1, x_9, x_0, x_{10}, x_3, y_1, y_9, y_0, y_{10}, y_3 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_1, x_9, x_0, x_{10}, x_3\} \cap \{y_1, y_9, y_0, y_{10}, y_3\} = \emptyset$$
$$P_{\beta,\alpha}(x_1, x_9, x_0, x_{10}, x_3)$$
$$P_{\beta,\alpha}(y_1, y_9, y_0, y_{10}, y_3)$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_1, x_9, x_0, x_{10}, x_3)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_9, y_0, y_{10}, y_3)$$

to see that

$$\text{if } \|S_\gamma(b)\| \geq 15,\text{then } M^0_{\beta,\alpha,\gamma}(x_0, x_{10}, x_3, x_1, x_9, y_0, y_{10}, y_3, y_1, y_9).(2)$$

Next, we note that $x_4 \overset{\delta}{\rightarrow} \text{——} y_4$ since $x_4, y_4 \in S_\beta(b) \cap S_\alpha(d) \cap S_\gamma(a)$. Note further that $c, y_6, y_4, y_8, a, x_8, x_4, x_6 \in S_\beta(b)$. Note also that the following hold:

$$\{y_6, y_4, y_8\} \cap \{x_8, x_4, x_6\} = \emptyset$$
$$P_{\alpha,\gamma}(c, y_6, y_4, y_8, a)$$
$$P_{\alpha,\gamma}(a, x_8, x_4, x_6, c)$$

We may now apply Lemma 12 with

$$(u_1, \ldots, u_8) = (c, y_6, y_4, y_8, a, x_8, x_4, x_6)$$

to see that

$$\begin{aligned} &\text{if } \|S_\beta(b)\| \geq 15, \text{ then } S_\beta(b) \text{ is twisted} \\ &\qquad \text{and either } N^2_{\delta,\gamma,\alpha}(c, y_6, y_4, y_8, a, x_8, x_4, x_6) \\ &\qquad \text{or } N^2_{\delta,\gamma,\alpha}(c, x_6, x_4, x_8, a, y_8, y_4, y_6). \end{aligned} \tag{3}$$

Next, we note that $x_5 \overset{\delta}{\rightarrow} \text{——} y_5$ since $x_5, y_5 \in S_\alpha(b) \cap S_\beta(c) \cap S_\gamma(d)$. Note further that $e, y_2, y_5, y_7, d, x_7, x_5, x_2 \in S_\alpha(b)$. Note also that the following hold:

$$\{y_2, y_5, y_7\} \cap \{x_7, x_5, x_2\} = \emptyset$$
$$P_{\beta,\gamma}(e, y_2, y_5, y_7, d)$$
$$P_{\beta,\gamma}(d, x_7, x_5, x_2, e)$$

We may now apply Lemma 12 with

$$(u_1, \ldots, u_8) = (e, y_2, y_5, y_7, d, x_7, x_5, x_2)$$

to see that

$$\begin{aligned} &\text{if } \|S_\alpha(b)\| \geq 15, \text{ then } S_\alpha(b) \text{ is twisted} \\ &\qquad \text{and either } N^2_{\delta,\gamma,\beta}(e, y_2, y_5, y_7, d, x_7, x_5, x_2) \\ &\qquad \text{or } N^2_{\delta,\gamma,\beta}(e, x_2, x_5, x_7, d, y_7, y_5, y_2). \end{aligned} \tag{4}$$

Next, we note that $x_3 \overset{\delta}{\rightarrow} \text{——} y_3$ since $x_3, y_3 \in S_\beta(a) \cap S_\alpha(c) \cap S_\gamma(b)$. Note further that $b, y_7, y_3, y_9, e, x_9, x_3, x_7 \in S_\beta(a)$. Note also that the following hold:

$$\{y_7, y_3, y_9\} \cap \{x_9, x_3, x_7\} = \emptyset$$
$$P_{\alpha,\gamma}(b, y_7, y_3, y_9, e)$$
$$P_{\alpha,\gamma}(e, x_9, x_3, x_7, b)$$

We may now apply Lemma 12 with

$$(u_1, \ldots, u_8) = (b, y_7, y_3, y_9, e, x_9, x_3, x_7)$$

to see that

$$\begin{aligned} &\text{if } \|S_\beta(a)\| \geq 15, \text{ then } S_\beta(a) \text{ is twisted} \\ &\qquad \text{and either } N^2_{\delta,\gamma,\alpha}(b, y_7, y_3, y_9, e, x_9, x_3, x_7) \\ &\qquad \text{or } N^2_{\delta,\gamma,\alpha}(b, x_7, x_3, x_9, e, y_9, y_3, y_7). \end{aligned} \tag{5}$$

From (1), we see that

$$\text{if } \|S_\gamma(a)\| \geq 15, \text{ then } x_0 \overset{\delta}{\rightarrow} \text{——} y_{10} \text{ and } x_6 \overset{\alpha}{\rightarrow} \text{——} y_4 \text{ and } x_4 \overset{\alpha}{\rightarrow} \text{——} y_6. (6)$$

From (2), we see that

$$\text{if } \|S_\gamma(b)\| \geq 15, \text{ then } x_0 \stackrel{\beta}{\to} \text{——}\, y_{10} \text{ and } x_9 \stackrel{\delta}{\to} \text{——}\, y_3 \text{ and } x_3 \stackrel{\delta}{\to} \text{——}\, y_9 .(7)$$

From (3), we see that

$$\text{if } \|S_\beta(b)\| \geq 15, \text{then } S_\beta(b) \text{ is twisted}$$

(8)
$$\text{and either } x_6 \stackrel{\delta}{\to} \text{——}\, y_4 \text{ or } x_4 \stackrel{\delta}{\to} \text{——}\, y_6 .$$

From (5), we see that

$$\text{if } \|S_\beta(a)\| \geq 15, \text{then } S_\beta(a) \text{ is twisted}$$

(9)
$$\text{and either } x_9 \stackrel{\alpha}{\to} \text{——}\, y_3 \text{ or } x_3 \stackrel{\alpha}{\to} \text{——}\, y_9 .$$

From (6) and (7), we see that

$$\text{either } \|S_\gamma(a)\| \leq 14 \text{ or } \|S_\gamma(b)\| \leq 14.(1)0$$

From (6) and (8), we see that

$$\text{either } \|S_\gamma(a)\| \leq 14 \text{ or } \|S_\beta(b)\| \leq 14.(1)1$$

From (7) and (9), we see that

$$\text{either } \|S_\gamma(b)\| \leq 14 \text{ or } \|S_\beta(a)\| \leq 14.(1)2$$

From (10) and (11), by Proposition 1, we see that

$$\|S_\gamma(a)\| \leq 14.(1)3$$

From (12) and (13), by Proposition 1, we see that

$$\|S_\gamma(b)\| \leq 14.(1)4$$

From (13) and (9), by Proposition 1, we see that

$$\|S_\gamma(a)\| \leq 14 \text{ and } S_\beta(a) \text{ is twisted.}(1)5$$

By an application of the symmetry Θ to (15), we see that

$$\|S_\gamma(a)\| \leq 14 \text{ and } S_\alpha(a) \text{ is twisted.}(1)6$$

From (15) and (16), we see that

$$\|S_\gamma(a)\| \leq 14 \text{ and both } S_\beta(a) \text{ and } S_\alpha(a) \text{ are twisted.}(1)7$$

From (14),(4), and (8), by Proposition 1, we see that

$$\|S_\gamma(b)\| \leq 14 \text{ and both } S_\beta(b) \text{ and } S_\alpha(b) \text{ are twisted.}(1)8$$

By an application of the symmetry Θ to (18), we see that

$$\|S_\gamma(c)\| \leq 14 \text{ and both } S_\alpha(c) \text{ and } S_\beta(c) \text{ are twisted.}(1)9$$

By an application of the symmetry Θ to (19), we see that

$$\|S_\gamma(e)\| \leq 14 \text{ and both } S_\beta(e) \text{ and } S_\alpha(e) \text{ are twisted.}(2)0$$

By an application of the symmetry Θ to (20), we see that

$$\|S_\gamma(d)\| \leq 14 \text{ and both } S_\alpha(d) \text{ and } S_\beta(d) \text{ are twisted.}(2)1$$

By (17), (18), (19),(20), and (21), we see that the proof is complete. □

Proposition 25. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^0_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto a$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_9 \longmapsto x_5 \longmapsto x_7 \longmapsto x_1$$
$$x_2 \longmapsto x_6 \longmapsto x_4 \longmapsto x_{10} \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1$$
$$y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

First, we note the following:

$$x_2 \xrightarrow{\delta} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(e).$$
$$x_{10} \xrightarrow{\delta} y_6 \text{ since } x_{10}, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$
$$x_6 \xrightarrow{\delta} y_{10} \text{ since } x_6, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$
$$x_4 \xrightarrow{\delta} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$
$$\text{Either } x_6 \xrightarrow{\alpha} y_2 \text{ or } x_6 \xrightarrow{\delta} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\beta(e).$$
$$\text{Either } x_{10} \xrightarrow{\alpha} y_4 \text{ or } x_{10} \xrightarrow{\delta} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\beta(b).$$
$$\text{Either } x_2 \xrightarrow{\alpha} y_{10} \text{ or } x_2 \xrightarrow{\delta} y_{10} \text{ since } x_2, y_{10} \in S_\gamma(a) \cap S_\beta(e).$$
$$\text{Either } x_4 \xrightarrow{\alpha} y_6 \text{ or } x_4 \xrightarrow{\delta} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\beta(b).$$
$$\text{Either } x_2 \xrightarrow{\beta} y_6 \text{ or } x_2 \xrightarrow{\delta} y_6 \text{ since } x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$
$$\text{Either } x_4 \xrightarrow{\beta} y_{10} \text{ or } x_4 \xrightarrow{\delta} y_{10} \text{ since } x_4, y_{10} \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_6 \xrightarrow{\beta} y_4 \text{ or } x_6 \xrightarrow{\delta} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_{10} \xrightarrow{\beta} y_2 \text{ or } x_{10} \xrightarrow{\delta} y_2 \text{ since } x_{10}, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$

Note further that $x_2, x_{10}, x_0, x_6, x_4, y_2, y_6, y_0, y_{10}, y_4 \in S_\gamma(a)$. Note also that the following hold:

$$\{x_2, x_{10}, x_0, x_6, x_4\} \cap \{y_2, y_6, y_0, y_{10}, y_4\} = \emptyset$$
$$P_{\beta,\alpha}(x_2, x_{10}, x_0, x_6, x_4)$$
$$P_{\beta,\alpha}(y_2, y_6, y_0, y_{10}, y_4)$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_2, x_{10}, x_0, x_6, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_2, y_6, y_0, y_{10}, y_4)$$

to see that

$$\text{if } \|S_\gamma(a)\| \geq 15, \text{ then } M^0_{\beta,\alpha,\delta}(x_2, x_{10}, x_0, x_6, x_4, y_2, y_6, y_0, y_{10}, y_4).(1)$$

Next, we note the following:

$$x_3 \overset{\delta}{\to} \text{——} y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(c) \cap S_\beta(a).$$
$$x_7 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_7, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(a).$$
$$x_6 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_6, y_{10} \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c).$$
$$x_5 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_5, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(c).$$
$$x_7 \overset{\delta}{\to} \text{——} y_1 \text{ since } x_7, y_1 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(d).$$
$$\text{Either } x_5 \overset{\alpha}{\to} \text{——} y_{10} \text{ or } x_5 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_5, y_{10} \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_7 \overset{\alpha}{\to} \text{——} y_3 \text{ or } x_7 \overset{\delta}{\to} \text{——} y_3 \text{ since } x_7, y_3 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_6 \overset{\alpha}{\to} \text{——} y_9 \text{ or } x_6 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_6, y_9 \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_3 \overset{\alpha}{\to} \text{——} y_9 \text{ or } x_3 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_3, y_9 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_5 \overset{\beta}{\to} \text{——} y_1 \text{ or } x_5 \overset{\delta}{\to} \text{——} y_1 \text{ since } x_5, y_1 \in S_\gamma(b) \cap S_\alpha(a).$$
$$\text{Either } x_6 \overset{\beta}{\to} \text{——} y_3 \text{ or } x_6 \overset{\delta}{\to} \text{——} y_3 \text{ since } x_6, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$
$$\text{Either } x_3 \overset{\beta}{\to} \text{——} y_{10} \text{ or } x_3 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d).$$

Note further that $x_3, x_5, x_7, x_0, x_6, y_3, y_1, y_9, y_0, y_{10} \in S_\gamma(b)$. Note also that the following hold:

$$\{x_3, x_5, x_7, x_0, x_6\} \cap \{y_3, y_1, y_9, y_0, y_{10}\} = \emptyset$$
$$P_{\beta,\alpha}(x_3, x_5, x_7, x_0, x_6)$$
$$P_{\beta,\alpha}(y_3, y_1, y_9, y_0, y_{10})$$

We may now apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_5, x_7, x_0, x_6)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_3, y_1, y_9, y_0, y_{10})$$

to see that

$$\text{if } \|S_\gamma(b)\| \geq 15, \text{ then } M^2_{\beta,\alpha,\gamma}(x_0, x_6, x_3, x_5, x_7, y_3, y_{10}, y_0, y_9, y_1).(2)$$

By an application of the symmetry Θ to (2), we see that

$$\text{if } \|S_\gamma(c)\| \geq 15, \text{ then } M^2_{\alpha,\beta,\delta}(x_0, x_4, x_8, x_7, x_1 y_8, y_4, y_0, y_1, y_7).(3)$$

By an application of the symmetry Θ to (3), we see that

$$\text{if } \|S_\gamma(e)\| \geq 15, \text{ then } M^2_{\beta,\alpha,\delta}(x_0, x_{10}, x_3, x_1, x_9, y_3, y_6, y_0, y_7, y_5).(4)$$

By an application of the symmetry Θ to (4), we see that

$$\text{if } \|S_\gamma(d)\| \geq 15, \text{ then } M^2_{\alpha,\beta,\delta}(x_0, x_2, x_8, x_9, x_5, y_8, y_2, y_0, y_5, y_9).(5)$$

Now, note that $c, y_6, y_4, y_8, a, x_8, x_4, x_{10} \in S_\beta(b)$. Note also that the following hold:

$$\{y_6, y_4, y_8\} \cap \{x_8, x_4, x_{10}\} = \emptyset$$
$$P_{\alpha,\gamma}(c, y_6, y_4, y_8, a)$$
$$P_{\alpha,\gamma}(a, x_8, x_4, x_{10}, c)$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (c, y_6, y_4, y_8, a, x_8, x_4, x_{10})$$

to see that

$$\text{if } x_{10} \overset{\alpha}{\to} \text{——} y_4 \text{ and } x_4 \overset{\alpha}{\to} \text{——} y_{10}, \text{ then } \|S_\beta(b)\| \le 14.(6)$$

Next, we note the following:

$$x_5 \overset{\delta}{\to} \text{——} y_5 \text{ since } x_5, y_5 \in S_\beta(c) \cap S_\alpha(a) \cap S_\gamma(b).$$

$$x_5 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_5, y_9 \in S_\beta(c) \cap S_\alpha(e) \cap S_\gamma(b).$$

Note further that $b, y_5, y_9, y_{10}, d, x_6, x_5, x_9 \in S_\beta(c)$. Note also that the following hold:

$$\{y_5, y_9, y_{10}\} \cap \{x_6, x_5, x_9\} = \emptyset$$
$$P_{\alpha,\gamma}(b, y_5, y_9, y_{10}, d)$$
$$P_{\alpha,\gamma}(d, x_6, x_5, x_9, b)$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (b, y_5, y_9, y_{10}, d, x_6, x_5, x_9)$$

to see that

$$\text{if } \|S_\beta(c)\| \ge 15, \text{ then } N^2_{\delta,\gamma,\alpha}(b, x_9, x_5, x_6, d, y_{10}, y_9, y_5).(7)$$

Next, we note the following:

$$x_1 \overset{\delta}{\to} \text{——} y_1 \text{ since } x_1, y_1 \in S_\beta(d) \cap S_\alpha(a) \cap S_\gamma(c).$$

$$x_1 \overset{\delta}{\to} \text{——} y_7 \text{ since } x_1, y_7 \in S_\beta(d) \cap S_\alpha(b) \cap S_\gamma(c).$$

Note further that $e, y_1, y_7, y_6, c, x_{10}, x_1, x_7 \in S_\beta(d)$. Note also that the following hold:

$$\{y_1, y_7, y_6\} \cap \{x_{10}, x_1, x_7\} = \emptyset$$
$$P_{\alpha,\gamma}(e, y_1, y_7, y_6, c)$$
$$P_{\alpha,\gamma}(c, x_{10}, x_1, x_7, e)$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (e, y_1, y_7, y_6, c, x_{10}, x_1, x_7)$$

to see that

$$\text{if } \|S_\beta(d)\| \ge 15, \text{ then } N^2_{\delta,\gamma,\alpha}(e, x_7, x_1, x_{10}, c, y_6, y_7, y_1).(8)$$

Finally, note that $b, y_7, y_3, y_9, e, x_7, x_3, x_9 \in S_\beta(a)$. note also that the following hold:

$$\{y_7, y_3, y_9\} \cap \{x_7, x_3, x_9\} = \emptyset$$
$$P_{\alpha,\gamma}(b, y_7, y_3, y_9, e)$$
$$P_{\alpha,\gamma}(e, x_7, x_3, x_9, b)$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (b, y_7, y_3, y_9, e, x_7, x_3, x_9)$$

to see that

$$\text{if } x_9 \overset{\alpha}{\to} \text{——} y_9 \text{ and } x_7 \overset{\alpha}{\to} \text{——} y_7, \text{ then } \|S_\beta(a)\| \le 14.(9)$$

Now, we show that $\|S_\gamma(a)\| \le 14$. Suppose not. Then $\|S_\gamma(a)\| \ge 15$, so that $x_0 \overset{\delta}{\to} \text{——} y_0$ by (1). But then $\|S_\gamma(b)\| \le 14$, since otherwise, we would have $x_0 \overset{\alpha}{\to} \text{——} y_0$ by (2). Since $\|S_\gamma(b)\| \le 14$, we must have $\|S_\beta(b)\| \geq 15$, by Proposition 1. But (1) implies that $x_{10} \overset{\alpha}{\to} —— y_4$ and $x_4 \overset{\alpha}{\to} —— y_{10}$, which gives a contradiction by (6). Thus, we have

$$\|S_\gamma(a)\| \leq 14, (1)0$$

as desired.

Next, we show that either $\|S_\gamma(c)\| \geq 15$ or $\|S_\gamma(d)\| \geq 15$ (or both). Suppose not. then $\|S_\gamma(c)\|, \|S_\gamma(d)\| \leq 14$. But this, in turn, implies that $\|S_\beta(c)\|, \|S_\beta(d)\| \geq 15, by Proposition$ 1.. Thus, we now have $x_9 \overset{\alpha}{\to} —— y_9$ by (7) and $x_7 \overset{\alpha}{\to} —— y_7$ by (8). But then (9) implies that $\|S_\beta(a)\| \leq 14$, which contradicts (10). Thus, we have

$$\text{either } \|S_\gamma(c)\| \geq 15 \text{ or } \|S_\gamma(d)\| \geq 15, (1)1$$

as desired.

By (11),(3), and (5), we have

$$x_0 \overset{\alpha}{\to} —— y_0.(1)2$$

By an application of the symmetry Θ to (12) we have

$$x_0 \overset{\beta}{\to} —— y_0.(1)3$$

(12) and (13) together produce the desired contradiction. The proof is complete. □

Proposition 26. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B^0_{\alpha,\beta,\gamma}(x_0, \dots, x_{15})$ and $B^0_{\alpha,\beta,\gamma}(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, $e = x_{12} = y_{12}$. If the graph has fewer than 62 vertices, then for any $w \in \{a, b, c, d, e\}$ we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted. (Note that both $\|S_\alpha(w)\|, \|S_\beta(w)\| \geq 15$ in this case, so that it makes sense to ask whether they are twisted in this last sentence.)*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto a$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_9 \longmapsto x_5 \longmapsto x_7 \longmapsto x_1$$
$$x_2 \longmapsto x_6 \longmapsto x_4 \longmapsto x_{10} \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_9 \longmapsto y_5 \longmapsto y_7 \longmapsto y_1$$
$$y_2 \longmapsto y_6 \longmapsto y_4 \longmapsto y_{10} \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Next, we note that the hypotheses of the proposition are preserved by the following symmetry Ψ

$$
\begin{gathered}
u \longmapsto u \\
v \longmapsto v \\
a \longmapsto b \longmapsto c \longmapsto d \longmapsto e \longmapsto a \\
x_0 \longmapsto x_0 \\
x_1 \longmapsto x_2 \longmapsto x_3 \longmapsto x_4 \longmapsto x_5 \longmapsto x_0 \\
x_6 \longmapsto x_7 \longmapsto x_8 \longmapsto x_9 \longmapsto x_{10} \longmapsto x_6 \\
y_0 \longmapsto y_0 \\
y_1 \longmapsto y_2 \longmapsto y_3 \longmapsto y_4 \longmapsto y_5 \longmapsto y_0 \\
y_6 \longmapsto y_7 \longmapsto y_8 \longmapsto y_9 \longmapsto y_{10} \longmapsto y_6
\end{gathered}
$$

Note that the symmetry Ψ acts on the colors as follows:

$$
\begin{gathered}
\alpha \longmapsto \alpha \\
\beta \longmapsto \beta \\
\gamma \longmapsto \gamma \\
\delta \longmapsto \delta
\end{gathered}
$$

We assume that the graph has at least 62 vertices.

First, we note the following:

$$
\begin{gathered}
x_2 \xrightarrow{\delta} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(e). \\
x_{10} \xrightarrow{\delta} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b). \\
x_6 \xrightarrow{\delta} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c). \\
x_4 \xrightarrow{\delta} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b). \\
\text{Either } x_6 \xrightarrow{\alpha} y_2 \text{ or } x_6 \xrightarrow{\delta} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\beta(e). \\
\text{Either } x_2 \xrightarrow{\alpha} y_6 \text{ or } x_2 \xrightarrow{\delta} y_6 \text{ since } x_2, y_6 \in S_\gamma(a) \cap S_\beta(e). \\
\text{Either } x_{10} \xrightarrow{\alpha} y_4 \text{ or } x_{10} \xrightarrow{\delta} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\beta(b). \\
\text{Either } x_4 \xrightarrow{\alpha} y_{10} \text{ or } x_4 \xrightarrow{\delta} y_{10} \text{ since } x_4, y_{10} \in S_\gamma(a) \cap S_\beta(b). \\
\text{Either } x_2 \xrightarrow{\beta} y_{10} \text{ or } x_2 \xrightarrow{\delta} y_{10} \text{ since } x_2, y_{10} \in S_\gamma(a) \cap S_\alpha(c). \\
\text{Either } x_{10} \xrightarrow{\beta} y_2 \text{ or } x_{10} \xrightarrow{\delta} y_2 \text{ since } x_{10}, y_2 \in S_\gamma(a) \cap S_\alpha(c). \\
\text{Either } x_6 \xrightarrow{\beta} y_4 \text{ or } x_6 \xrightarrow{\delta} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\alpha(d). \\
\text{Either } x_4 \xrightarrow{\beta} y_6 \text{ or } x_4 \xrightarrow{\delta} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(d).
\end{gathered}
$$

Note further that $x_2, x_{10}, x_0, x_6, x_4, y_2, y_{10}, y_0, y_6, y_4 \in S_\gamma(a)$. note also that the following hold:

$$
\begin{gathered}
\{x_2, x_{10}, x_0, x_6, x_4\} \cap \{y_2, y_{10}, y_0, y_6, y_4\} = \emptyset \\
P_{\beta,\alpha}(x_2, x_{10}, x_0, x_6, x_4) \\
P_{\beta,\alpha}(y_2, y_{10}, y_0, y_6, y_4)
\end{gathered}
$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_2, x_{10}, x_0, x_6, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_2, y_{10}, y_0, y_6, y_4)$$

to see that

$$\text{if } \|S_\gamma(a)\| \geq 15, \text{ then } M^2_{\beta,\alpha,\delta}(x_2, x_{10}, x_0, x_6, x_4, y_2, y_{10}, y_0, y_6, y_4). (1)$$

Now, note that $x_6 \xrightarrow{\delta} y_6$ since $x_6, y_6 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(a)$. Note further that $a, y_3, y_6, y_4, b, x_4, x_6, x_3 \in S_\alpha(d)$. Note also that the following hold:

$$\{y_3, y_6, y_4,\} \cap \{x_4, x_6, x_3\} = \emptyset$$
$$P_{\beta,\gamma}(a, y_3, y_6, y_4, b)$$
$$P_{\beta,\gamma}(b, x_4, x_6, x_3, a)$$

We may now apply Lemma 9 with

$$(u_1, \ldots, u_8) = (a, y_3, y_6, y_4, b, x_4, x_6, x_3)$$

to see that

$$\begin{aligned} &\text{if } \|S_\alpha(d)\| \geq 15, \text{ then } S_\alpha(d) \text{ is twisted} \\ &\quad \text{and either } N^2_{\delta,\gamma,\beta}(a, y_3, y_6, y_4, b, x_4, x_6, x_3) \\ &\quad \text{or } N^2_{\delta,\gamma,\beta}(a, x_3, x_6, x_4, b, y_4, y_6, y_3). \end{aligned} \tag{2}$$

By (2), we see that

$$\begin{aligned} &\text{if } \|S_\alpha(d)\| \geq 15, \text{then } S_\alpha(d) \text{ is twisted} \\ &\quad \text{and either } x_6 \xrightarrow{\delta} y_4 \text{ or } x_4 \xrightarrow{\delta} y_6. \end{aligned} \tag{3}$$

Suppose that $\|S_\gamma(a)\| \geq 15$. Then, by (1), we have $x_6 \xrightarrow{\beta} y_4$ and $x_4 \xrightarrow{\beta} y_6$. By (3), this implies that $\|S_\alpha(d)\| \leq 14$. Thus, we have shown that

$$\text{if } \|S_\gamma(a)\| \geq 15, \text{ then } \|S_\alpha(d)\| \leq 14. (4)$$

From (4), by Proposition 1, we see that

$$\text{if } \|S_\gamma(a)\| \geq 15, \text{ then } \|S_\gamma(d)\| \geq 15. (5)$$

By an application of the symmetry Θ to (4), we see that

$$\text{if } \|S_\gamma(a)\| \geq 15, \text{ then } \|S_\beta(e)\| \leq 14. (6)$$

By an application of the symmetry Ψ to (4), we see that

$$\text{if } \|S_\gamma(b)\| \geq 15, \text{ then } \|S_\alpha(e)\| \leq 14. (7)$$

Repeated applications of the symmetry Ψ to (5) shows that either $\|S_\gamma(w)\| \geq 15$ for all $w \in \{a, b, c, d, e\}$ or else $\|S_\gamma(w)\| \leq 14$ for all $w \in \{a, b, c, d, e\}$. Suppose that $\|S_\gamma(w)\| \geq 15$ for all $w \in \{a, b, c, d, e\}$. Then by (7), we have $\|S_\alpha(e)\| \leq 14$. But (6) implies that $\|S_\beta(e)\| \leq 14$, which is a contradiction. Thus, we have

$$\|S_\gamma(w)\| \geq 15 \text{ for all } w \in \{a, b, c, d, e\}. (8)$$

By (8) and (2), we see that

$$S_\alpha(d) \text{ is twisted}. (9)$$

By repeated applications of the symmetry Ψ to (9), we see that

$$S_\alpha(w) \text{ is twisted for all } w \in \{a, b, c, d, e\}. (1)0$$

by an application of the symmetry θ to (10), we see that

$$S_\beta(w) \text{ is twisted for all } w \in \{a,b,c,d,e\}.(1)1$$

By (8),(10),and (11), we see that the proof is complete. □

Theorem 11. *Let V be the vertex set of a complete graph with a good edge coloring with four colors. Suppose that $\|V\| = 62$ and let $u, v \in V$ be such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ and $\|S_\delta(u) \cap S_\delta(v)\| = 5$, for some color δ. Then there exist $x_0, \ldots, x_{15} \in S_\delta(u)$ and $y_0, \ldots, x_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{11, 12, 13, 14, 15\}$ and some $j \in \{0,1\}$ and colors α, β, and γ, such that $B^j_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^j_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ with*

$$S_\delta(u) \cap S_\delta(v) = \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\} = \{y_{11}, y_{12}, y_{13}, y_{14}, y_{15}\}.$$

Furthermore, for each $w \in S_\delta(u) \cap S_\delta(v)$, we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted.

Proof. Since $\|S_\delta(u) \cap S_\delta(v)\| = 5$, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{w_0, w_1, w_2, w_3, w_4\}.$$

By Proposition 22, we may suppose, without loss of generality, that

$$w_0 \stackrel{\alpha}{\to} \text{---} w_1, w_4(1)$$

and

$$w_0 \stackrel{\beta}{\to} \text{---} w_2, w_3.(2)$$

By (1), we cannot have $w_1 \stackrel{\alpha}{\to} \text{---} w_4$, since otherwise we would have a monochromatic triangle. Thus, since $w_1 \stackrel{\alpha}{\to} \text{---} w_4$ by (1), we see, by Proposition 22, that either $w_1 \stackrel{\alpha}{\to} \text{---} w_2$ or $w_1 \stackrel{\alpha}{\to} \text{---} w_3$. Without loss of generality, we may suppose that

$$w_1 \stackrel{\alpha}{\to} \text{---} w_2.(3)$$

By (2), we cannot have $w_2 \stackrel{\beta}{\to} \text{---} w_3$, since otherwise we would have a monochromatic triangle. Thus, since $w_2 \stackrel{\alpha}{\to} \text{---} w_1$ by (3) and $w_2 \stackrel{\beta}{\to} \text{---} w_0$ by (2), we see, by Proposition 22, that

$$w_2 \stackrel{\alpha}{\to} \text{---} w_3(4)$$

and

$$w_2 \stackrel{\beta}{\to} \text{---} w_4.(5)$$

By (1), we cannot have $w_4 \stackrel{\alpha}{\to} \text{---} w_1$, since otherwise we would have a monochromatic triangle. Thus, since $w_4 \stackrel{\alpha}{\to} \text{---} w_0$ by (1) and $w_4 \stackrel{\beta}{\to} \text{---} w_2$ by (5), we see, by Proposition 22, that

$$w_4 \stackrel{\beta}{\to} \text{---} w_1(6)$$

and

$$w_4 \stackrel{\alpha}{\to} \text{---} w_3.(7)$$

Since $w_1 \stackrel{\alpha}{\to} \text{---} w_0, w_2$ by (1) and (3), and $w_1 \stackrel{\beta}{\to} \text{---} w_4$ by (6), we see, by Proposition 22, that

$$w_1 \stackrel{\beta}{\to} \text{---} w_3.(8)$$

Thus, we see, by (1), (2), (3), (4), (5), (6), (7), and (8), that

$$P_{\alpha,\beta}(w_0, w_1, w_2, w_3, w_4).$$

The conclusion now follows by Proposition 23, Proposition 24, Proposition 25, and Proposition 26.
The proof is complete. □

4.12. **3.12. Attaching Sets of Cardinality** 4.

Proposition 27. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 4$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned}(51)\quad &\big(\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\|\big)\\ &\qquad\qquad \in [3,0,0]\cup[2,1,0].\end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u)\cap S_\delta(v)\cap S_\eta(x)\big\| \le R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u)\cap S_\delta(v)\cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u)\cap S_\delta(v)\cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned}(52)\quad S_\delta(u)\cap S_\delta(v) = \{x\} &\uplus \big(S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\big)\\ &\uplus \big(S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\big) \uplus \big(S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\big),\end{aligned}$$

so that

$$\begin{aligned}(53)\quad 4 = \big\|S_\delta(u)\cap S_\delta(v)\big\| &= 1 + \big\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\big\|\\ &+ \big\|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\big\| + \big\|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\big\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}(54)\quad &\big(\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\|\big)\\ &\qquad\qquad \in [3,0,0]\cup[2,1,0]\cup[1,1,1].\end{aligned}$$

The proposition now follows by an application of Proposition 4. ☐

Proposition 28. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 4$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned}(55)\quad &\big(\|S_\delta(u)\cap S_\delta(v)\cap S_\alpha(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\beta(x)\|, \|S_\delta(u)\cap S_\delta(v)\cap S_\gamma(x)\|\big)\\ &\qquad\qquad \in [2,1,0].\end{aligned}$$

Proof. Suppose not. Then, by Proposition 27, we may suppose, without loss of generality, that $S_\delta(u)\cap S_\delta(v) = \{x, w_0, w_1, w_2\}$ with $x \overset{\alpha}{\to}$ —— w_0, w_1, w_2. We cannot have $w_0 \overset{\alpha}{\to}$ —— w_i for any $i \in \{1,2\}$, since otherwise we would have a monochromatic triangle. Since $w_0 \overset{\alpha}{\to}$ —— x, we may, by Proposition 27, suppose, without loss of generality, that $w_0 \overset{\beta}{\to}$ —— w_1, w_2. Now, we must have $w_1 \overset{\gamma}{\to}$ —— w_2 , since $w_1, w_2 \in S_\alpha(x)\cap S_\beta(w_0)\cap S_\delta(u)$. Thus, we have $w_1 \overset{\alpha}{\to}$ —— x and $w_1 \overset{\beta}{\to}$ —— w_0 and $w_1 \overset{\gamma}{\to}$ —— w_2, which contradicts Proposition 27.

The proof is complete. ☐

Proposition 29. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u)\cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_0 = y_0$, $b = x_1 = y_1$, $c = x_{12} = y_{12}$, and $d = x_4 = y_4$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto b \longmapsto c \longmapsto d \longmapsto a$$
$$x_2 \longmapsto x_{13} \longmapsto x_9 \longmapsto x_6 \longmapsto x_2$$
$$x_3 \longmapsto x_7 \longmapsto x_{14} \longmapsto x_{11} \longmapsto x_3$$
$$x_5 \longmapsto x_{10} \longmapsto x_{15} \longmapsto x_8 \longmapsto x_5$$
$$y_2 \longmapsto y_{13} \longmapsto y_9 \longmapsto y_6 \longmapsto y_2$$
$$y_3 \longmapsto y_7 \longmapsto y_{14} \longmapsto y_{11} \longmapsto y_3$$
$$y_5 \longmapsto y_{10} \longmapsto y_{15} \longmapsto y_8 \longmapsto y_5$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \alpha$$
$$\beta \longmapsto \gamma \longmapsto \beta$$
$$\delta \longmapsto \delta$$

Next, we note that the hypotheses of the proposition are preserved by the following symmetry Ψ:

$$u \longmapsto v \longmapsto u$$
$$a \longmapsto a$$
$$b \longmapsto b$$
$$c \longmapsto c$$
$$d \longmapsto d$$
$$x_i \longmapsto y_i \longmapsto x_i \quad \text{for all} \quad i \in \{0, \ldots, 15\}$$

Note that the symmetry Ψ acts on the colors as follows:

$$\alpha \longmapsto \alpha$$
$$\beta \longmapsto \beta$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\gamma(a) = \{x_{11}, c, x_{13}, x_{14}, x_{15}\} \subseteq S_\gamma(a)$ and $S_\delta(v) \cap S_\gamma(a) = \{y_{11}, c, y_{13}, y_{14}, y_{15}\} \subseteq S_\gamma(a)$. Note further that both $P_{\beta,\alpha}(x_{11}, c, x_{13}, x_{14}, x_{15})$ and $P_{\beta,\alpha}(y_{11}, c, y_{13}, y_{14}, y_{15})$ hold and that

$$\|\{x_{11}, c, x_{13}, x_{14}, x_{15}\} \cap \{y_{11}, c, y_{13}, y_{14}, y_{15}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\gamma(a)\| \leq 14$ so that

$$\|S_\alpha(a)\|, \|S_\beta(a)\| \geq 15. (1)$$

Repeated applications of the symmetry Θ to (1) give

$$\|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15, (2)$$
$$\|S_\alpha(c)\|, \|S_\beta(c)\| \geq 15, (3)$$

and

$$\|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15. (4)$$

Now, we note the following:

$$x_6 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_6 \text{ since } x_6, y_6 \in S_\beta(a) \cap S_\alpha(d) \cap S_\gamma(b).$$
$$x_9 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_9 \text{ since } x_9, y_9 \in S_\beta(a) \cap S_\alpha(c) \cap S_\gamma(d).$$
$$x_7 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_7 \text{ since } x_7, y_7 \in S_\beta(a) \cap S_\alpha(b) \cap S_\gamma(c).$$
$$\text{Either } x_{10} \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_{10} \text{ or } x_{10} \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_{10} \text{ since } x_{10}, y_{10} \in S_\beta(a) \cap S_\alpha(b).$$
$$\text{Either } x_8 \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_8 \text{ or } x_8 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_8 \text{ since } x_8, y_8 \in S_\beta(a) \cap S_\alpha(d).$$
$$\text{Either } x_7 \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_{10} \text{ or } x_7 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_{10} \text{ since } x_7, y_{10} \in S_\beta(a) \cap S_\alpha(b).$$
$$\text{Either } x_{10} \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_7 \text{ or } x_{10} \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_7 \text{ since } x_{10}, y_7 \in S_\beta(a) \cap S_\alpha(b).$$
$$\text{Either } x_6 \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_8 \text{ or } x_6 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_8 \text{ since } x_6, y_8 \in S_\beta(a) \cap S_\alpha(d).$$
$$\text{Either } x_8 \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_6 \text{ or } x_8 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_6 \text{ since } x_8, y_6 \in S_\beta(a) \cap S_\alpha(d).$$
$$\text{Either } x_6 \stackrel{\alpha}{\to} \!\!\!\!\text{——}\; y_7 \text{ or } x_6 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_7 \text{ since } x_6, y_7 \in S_\beta(a) \cap S_\gamma(c).$$
$$\text{Either } x_7 \stackrel{\alpha}{\to} \!\!\!\!\text{——}\; y_6 \text{ or } x_7 \stackrel{\delta}{\to} \!\!\!\!\text{——}\; y_6 \text{ since } x_7, y_6 \in S_\beta(a) \cap S_\gamma(c).$$

Note further that $x_8, x_6, x_9, x_7, x_{10}, y_8, y_6, y_9, y_7, y_{10} \in S_\beta(a)$. Note also that the following hold:

$$\{x_8, x_6, x_9, x_7, x_{10}\} \cap \{y_8, y_6, y_9, y_7, y_{10}\} = \emptyset$$
$$P_{\gamma,\alpha}(x_8, x_6, x_9, x_7, x_{10})$$
$$P_{\gamma,\alpha}(y_8, y_6, y_9, y_7, y_{10})$$

Since $\|S_\beta(a)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_8, x_6, x_9, x_7, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_8, y_6, y_9, y_7, y_{10})$$

to see that one of the following six possibilities must hold:

(1) "(a.i)" $M^0_{\gamma,\alpha,\delta}(x_9, x_7, x_{10}, x_8, x_6, y_9, y_7, y_{10}, y_8, y_6)$
(2) "(a.ii)" $M^0_{\gamma,\alpha,\delta}(x_7, x_{10}, x_8, x_6, x_9, y_7, y_{10}, y_8, y_6, y_9)$
(3) "(a.iii)" $M^2_{\gamma,\alpha,\delta}(x_{10}, x_7, x_9, x_6, x_8, y_{10}, y_7, y_9, y_6, y_8)$
(4) "(a.iv)" $M^2_{\gamma,\alpha,\delta}(x_8, x_6, x_9, x_7, x_{10}, y_8, y_6, y_9, y_7, y_{10})$
(5) "(a.v)" $M^2_{\gamma,\alpha,\delta}(x_9, x_6, x_8, x_{10}, x_7, y_6, y_8, y_{10}, y_7, y_9)$
(6) "(a.vi)" $M^2_{\gamma,\alpha,\delta}(x_9, x_7, x_{10}, x_8, x_6, y_7, y_{10}, y_8, y_6, y_9)$

Now, we show that (a.i) must fail. Suppose not. Then (a.i) holds, so that $x_8 \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_6$ and $x_6 \stackrel{\gamma}{\to} \!\!\!\!\text{——}\; y_8$. Note that $a, y_{11}, y_8, y_6, c, x_6, x_8, x_{11} \in S_\alpha(d)$. Note also that the following hold:

$$\{y_{11}, y_8, y_6\} \cap \{x_6, x_8, x_{11}\} = \emptyset$$
$$P_{\gamma,\beta}(a, y_{11}, y_8, y_6, c)$$
$$P_{\gamma,\beta}(c, x_6, x_8, x_{11}, a)$$

Since $\|S_\alpha(d)\| \geq 15$ by (4), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_{11}, y_8, y_6, c, x_6, x_8, x_{11})$$

to get a contradiction. Thus, (a.i) must fail.

Now, we show that (a.ii) must fail. Suppose not. Then (a.ii) holds, so that $x_7 \overset{\gamma}{\rightarrow}$ —— y_{10} and $x_{10} \overset{\gamma}{\rightarrow}$ —— y_7. Note that $a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13} \in S_\alpha(b)$. Note also that the following hold:

$$\{y_{13}, y_{10}, y_7\} \cap \{x_7, x_{10}, x_{13}\} = \emptyset$$
$$P_{\gamma,\beta}(a, y_{13}, y_{10}, y_7, c)$$
$$P_{\gamma,\beta}(c, x_7, x_{10}, x_{13}, a)$$

Since $\|S_\alpha(b)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13})$$

to get a contradiction. Thus, (a.ii) must fail.

Since both (a.i) and (a.ii) fail, one of (a.iii), (a.iv), (a.v), or (a.vi) must hold. An application of the symmetry Θ to this says that one of (b.iii), (b.iv), (b.v), or (b.vi) must hold.

(1) "(b.iii)" $M^2_{\beta,\alpha,\delta}(x_{15}, x_{14}, x_6, x_2, x_5, y_{15}, y_{14}, y_6, y_2, y_5)$
(2) "(b.iv)" $M^2_{\beta,\alpha,\delta}(x_5, x_2, x_6, x_{14}, x_{15}, y_5, y_2, y_6, y_{14}, y_{15})$
(3) "(b.v)" $M^2_{\beta,\alpha,\delta}(x_6, x_2, x_5, x_{15}, x_{14}, y_2, y_5, y_{15}, y_{14}, y_6)$
(4) "(b.vi)" $M^2_{\beta,\alpha,\delta}(x_6, x_{14}, x_{15}, x_5, x_2, y_{14}, y_{15}, y_5, y_2, y_6)$

An application of the symmetry Θ to this says that one of (c.iii), (c.iv), (c.v), or (c.vi) must hold.

(1) "(c.iii)" $M^2_{\gamma,\alpha,\delta}(x_8, x_{11}, x_2, x_{13}, x_{10}, y_8, y_{11}, y_2, y_{13}, y_{10})$
(2) "(c.iv)" $M^2_{\gamma,\alpha,\delta}(x_{10}, x_{13}, x_2, x_{11}, x_8, y_{10}, y_{13}, y_2, y_{11}, y_8)$
(3) "(c.v)" $M^2_{\gamma,\alpha,\delta}(x_2, x_{13}, x_{10}, x_8, x_{11}, y_{13}, y_{10}, y_8, y_{11}, y_2)$
(4) "(c.vi)" $M^2_{\gamma,\alpha,\delta}(x_2, x_{11}, x_8, x_{10}, x_{13}, y_{11}, y_8, y_{10}, y_{13}, y_2)$

An application of the symmetry Θ to this says that one of (d.iii), (d.iv), (d.v), or (d.vi) must hold.

(1) "(d.iii)" $M^2_{\beta,\alpha,\delta}(x_5, x_3, x_{13}, x_9, x_{15}, y_5, y_3, y_{13}, y_9, y_{15})$
(2) "(d.iv)" $M^2_{\beta,\alpha,\delta}(x_{15}, x_9, x_{13}, x_3, x_5, y_5, y_3, y_{13}, y_9, y_{15})$
(3) "(d.v)" $M^2_{\beta,\alpha,\delta}(x_{13}, x_9, x_{15}, x_5, x_3, y_9, y_{15}, y_5, y_3, y_{13})$
(4) "(d.vi)" $M^2_{\beta,\alpha,\delta}(x_{13}, x_3, x_5, x_{15}, x_9, y_3, y_5, y_{15}, y_9, y_{13})$

Now, we show that (b.iii) must fail. Suppose not. Then (b.iii) holds. If (c.iii) holds, then $x_{11} \overset{\alpha}{\rightarrow}$ —— y_2. But $x_{14} \overset{\alpha}{\rightarrow}$ —— y_2 by (b.iii), which gives a monochromatic triangle since $x_{11} \overset{\alpha}{\rightarrow}$ —— x_{14}. Thus, (c.iii) fails. If (c.iv) holds, then $x_2 \overset{\alpha}{\rightarrow}$ —— y_{11}. But $x_2 \overset{\alpha}{\rightarrow}$ —— y_14 by (b.iii), which gives a monochromatic triangle since $y_{11} \overset{\alpha}{\rightarrow}$ —— y_{14}. Thus, (c.iv) fails. If (c.v) holds, then $x_{10} \overset{\alpha}{\rightarrow}$ —— y_2. But $x_6 \overset{\alpha}{\rightarrow}$ —— y_2 by (b.iii), which gives a monochromatic triangle since $x_6 \overset{\alpha}{\rightarrow}$ —— x_{10}. Thus, (c.v) fails. Therefore, we conclude that (c.vi) must hold. If (d.iii) holds, then $x_3 \overset{\alpha}{\rightarrow}$ —— y_{13}. But $x_{11} \overset{\alpha}{\rightarrow}$ —— y_{13} by (c.vi), which gives a monochromatic triangle since $x_{11} \overset{\alpha}{\rightarrow}$ —— y_3. Thus, (d.iii) fails. If (d.iv) holds, then $x_9 \overset{\alpha}{\rightarrow}$ —— y_{13}. But $x_8 \overset{\alpha}{\rightarrow}$ —— y_{13} by (c.vi), which gives a monochromatic triangle since $x_9 \overset{\alpha}{\rightarrow}$ —— x_8. Thus (d.iv) fails. Therefore, we conclude that either (d.v) or (d.vi) must hold. But then $x_5 \overset{\beta}{\rightarrow}$ —— y_5, which is a contradiction, since $x_5 \overset{\delta}{\rightarrow}$ —— y_5 by (b.iii). Thus, we have shown that (b.iii) fails, as desired.

Note that $M^2_{\beta,\alpha,\gamma}(y_{15}, y_{14}, y_6, y_2, y_5, x_{15}, x_{14}, x_6, x_2, x_5)$ must also fail, by an application of the symmetry Ψ. It is trivial to see, directly from Definition 2(4), that $M^2_{\beta,\alpha,\gamma}(y_{15}, y_{14}, y_6, y_2, y_5, x_{15}, x_{14}, x_6, x_2, x_5)$ is equivalent to (b.iv). Thus, we have shown that (b.iv) fails, also. That is, both (b.iii) and (b.iv) must fail. Thus, we have

either (b.v) holds or (b.vi) holds.(5)

Repeated applications of the symmetry Θ give

either (c.v) holds or (c.vi) holds,(6)

$$\text{either (d.v) holds or (d.vi) holds,}\tag{7}$$

and

$$\text{either (a.v) holds or (a.vi) holds.}\tag{8}$$

Now, suppose that (c.v) holds. Then we must have $x_{10} \overset{\gamma}{\rightarrow} — y_{10}$, by (c.v). Also, $x_{13} \overset{\gamma}{\rightarrow} — y_2$ by (c.v) , and $y_2 \overset{\gamma}{\rightarrow} — y_7$ so that the edge from x_{13} and y_7 cannot be of color γ, since otherwise we would have a monochromatic triangle. Note that $a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13} \in S_\alpha(b)$. Note also that the following hold:

$$\{y_{13}, y_{10}, y_7\} \cap \{x_7, x_{10}, x_{13}\} = \emptyset$$
$$P_{\gamma,\beta}(a, y_{13}, y_{10}, y_7, c)$$
$$P_{\gamma,\beta}(c, x_7, x_{10}, x_{13}, a)$$

Since $\|S_\alpha(b)\| \geq 15$ by (2), We may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13})$$

to get

$$N^1_{\delta,\beta,\gamma}(a, x_{13}, x_{10}, x_7, c, y_7, y_{10}, y_{13}).$$

But then $x_{13} \overset{\beta}{\rightarrow} — y_7$. If (d.vi) holds, then $x_{13} \overset{\beta}{\rightarrow} — y_5$, which gives a monochromatic triangle, since $y_5 \overset{\beta}{\rightarrow} — y_7$. Thus, (d.vi) fails, so that by (7), (d.v) must hold. Thus, we have shown that

$$\text{if (c.v) holds, then (d.v) holds.}\tag{9}$$

By an application of the symmetry Θ to (9), we see that

$$\text{if (d.v) holds, then (a.v) holds.}\tag{10}$$

Now, we show that (c.v) must fail. Suppose not. Then (c.v) holds. By (9) and (10), (a.v) also holds. But $x_5 \overset{\delta}{\rightarrow} — y_{15}$ by (c.v), whereas $x_5 \overset{\alpha}{\rightarrow} — y_{15}$ by (a.v), which is a contradiction. Thus, we have shown that

$$\text{(c.v) fails,}\tag{11}$$

as desired.

By (6) and (11), we see that

$$\text{(c.vi) holds.}\tag{12}$$

The fact that $M^2_{\gamma,\alpha,\delta}(y_2, y_{13}, y_{10}, y_8, y_{11}, x_{13}, x_{10}, x_8, x_{11}, x_2)$ is equivalent to (c.vi) follows trivially from Definition 2(4). Thus, we have

$$M^2_{\gamma,\alpha,\delta}(y_2, y_{13}, y_{10}, y_8, y_{11}, x_{13}, x_{10}, x_8, x_{11}, x_2)\tag{13}$$

by (12). By an application of the symmetry Ψ to (13) we see that

$$\text{(c.v) holds.}\tag{14}$$

Thus, we have the desired contradiction, by (11) and (14). The proof is complete. □

Proposition 30. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_0 = y_0$, $b = x_1 = y_4$, $c = x_{12} = y_{12}$, and $d = x_4 = y_1$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto b \longmapsto c \longmapsto d \longmapsto a$$
$$x_2 \longmapsto x_{13} \longmapsto x_9 \longmapsto x_6 \longmapsto x_2$$
$$x_3 \longmapsto x_7 \longmapsto x_{14} \longmapsto x_{11} \longmapsto x_3$$
$$x_5 \longmapsto x_{10} \longmapsto x_{15} \longmapsto x_8 \longmapsto x_5$$
$$y_2 \longmapsto y_6 \longmapsto y_9 \longmapsto y_{13} \longmapsto y_2$$
$$y_3 \longmapsto y_{11} \longmapsto y_{14} \longmapsto y_7 \longmapsto y_3$$
$$y_5 \longmapsto y_8 \longmapsto y_{15} \longmapsto y_{10} \longmapsto y_5$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \alpha$$
$$\beta \longmapsto \gamma \longmapsto \beta$$
$$\delta \longmapsto \delta$$

Next, we note that the hypotheses of the proposition are preserved by the following symmetry Ψ:

$$u \longmapsto v \longmapsto u$$
$$a \longmapsto a$$
$$b \longmapsto d \longmapsto b$$
$$c \longmapsto c$$
$$x_i \longmapsto y_i \longmapsto x_i \quad \text{for all} \quad i \in \{0, \dots, 15\}$$

Note that the symmetry Ψ acts on the colors as follows:

$$\alpha \longmapsto \alpha$$
$$\beta \longmapsto \beta$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\gamma(a) = \{x_{11}, c, x_{13}, x_{14}, x_{15}\} \subseteq S_\gamma(a)$ and $S_\delta(v) \cap S_\gamma(a) = \{y_{11}, c, y_{13}, y_{14}, y_{15}\} \subseteq S_\gamma(a)$. Note further that both $P_{\beta,\alpha}(x_{11}, c, x_{13}, x_{14}, x_{15})$ and $P_{\beta,\alpha}(y_{11}, c, y_{13}, y_{14}, y_{15})$ hold and that

$$\|\{x_{11}, c, x_{13}, x_{14}, x_{15}\} \cap \{y_{11}, c, y_{13}, y_{14}, y_{15}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\gamma(a)\| \leq 14$ so that

$$\|S_\alpha(a)\|, \|S_\beta(a)\| \geq 15.(1)$$

Repeated applications of the symmetry Θ to (1) give

$$\|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15, (2)$$
$$\|S_\alpha(c)\|, \|S_\beta(c)\| \geq 15, (3)$$

and

$$\|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15.(4)$$

Note that either $x_2 \overset{\gamma}{\to} \!\!\longrightarrow y_2$ or $x_2 \overset{\delta}{\to} \!\!\longrightarrow y_2$ since $x_2, y_2 \in S_\alpha(a) \cap S_\beta(c)$. Note further that $d, y_3, y_5, y_2, b, x_3, x_5, x_2 \in S_\alpha(a)$. Note also that the following hold:

$$\{y_3, y_5, y_2\} \cap \{x_3, x_5, x_2\} = \emptyset$$
$$P_{\beta,\gamma}(d, y_3, y_5, y_2, b)$$
$$P_{\beta,\gamma}(b, x_3, x_5, x_2, d)$$

Since $\|S_\alpha(a)\| \geq 15$ by (1), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (d, y_3, y_5, y_2, b, x_3, x_5, x_2)$$

to see that

$$\begin{aligned} &\text{either } N^1_{\delta,\gamma,\beta}(d, x_2, x_5, x_3, b, y_2, y_5, y_3) \\ &\quad\text{or } N^2_{\delta,\gamma,\beta}(d, x_2, x_5, x_3, b, y_2, y_5, y_3). \end{aligned} \tag{5}$$

By repeated applications of the symmetry Θ, we see that

$$\begin{aligned} &\text{either } N^1_{\delta,\beta,\gamma}(a, x_{13}, x_{10}, x_7, c, y_6, y_8, y_{11}) \\ &\quad\text{or } N^2_{\delta,\beta,\gamma}(a, x_{13}, x_{10}, x_7, c, y_6, y_8, y_{11}), \end{aligned} \tag{6}$$

$$\begin{aligned} &\text{either } N^1_{\delta,\gamma,\beta}(b, x_9, x_{15}, x_{14}, d, y_9, y_{15}, y_{14}) \\ &\quad\text{or } N^2_{\delta,\gamma,\beta}(b, x_9, x_{15}, x_{14}, d, y_9, y_{15}, y_{14}), \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\text{either } N^1_{\delta,\beta,\gamma}(c, x_6, x_8, x_{11}, a, y_{13}, y_{10}, y_7) \\ &\quad\text{or } N^2_{\delta,\beta,\gamma}(c, x_6, x_8, x_{11}, a, y_{13}, y_{10}, y_7). \end{aligned} \tag{8}$$

Now, we note the following:

$$x_2 \overset{\delta}{\to} \!\!\longrightarrow y_3 \text{ by (5).}$$
$$x_{14} \overset{\delta}{\to} \!\!\longrightarrow y_{15} \text{ by (7).}$$
$$x_2 \overset{\delta}{\to} \!\!\longrightarrow y_3 \text{ since } x_2, y_3 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(d).$$
$$x_{14} \overset{\delta}{\to} \!\!\longrightarrow y_9 \text{ since } x_{14}, y_9 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(d).$$
$$\text{Either } x_2 \overset{\alpha}{\to} \!\!\longrightarrow y_9 \text{ or } x_2 \overset{\delta}{\to} \!\!\longrightarrow y_9 \text{ since } x_2, y_9 \in S_\gamma(b) \cap S_\beta(d).$$
$$\text{Either } x_2 \overset{\alpha}{\to} \!\!\longrightarrow y_{13} \text{ or } x_2 \overset{\delta}{\to} \!\!\longrightarrow y_{13} \text{ since } x_2, y_{13} \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_{14} \overset{\alpha}{\to} \!\!\longrightarrow y_3 \text{ or } x_{14} \overset{\delta}{\to} \!\!\longrightarrow y_3 \text{ since } x_{14}, y_3 \in S_\gamma(b) \cap S_\beta(d).$$
$$\text{Either } x_6 \overset{\alpha}{\to} \!\!\longrightarrow y_9 \text{ or } x_6 \overset{\delta}{\to} \!\!\longrightarrow y_9 \text{ since } x_6, y_9 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_2 \overset{\beta}{\to} \!\!\longrightarrow y_5 \text{ or } x_2 \overset{\delta}{\to} \!\!\longrightarrow y_5 \text{ since } x_2, y_5 \in S_\gamma(b) \cap S_\alpha(a).$$
$$\text{Either } x_5 \overset{\beta}{\to} \!\!\longrightarrow y_5 \text{ or } x_5 \overset{\delta}{\to} \!\!\longrightarrow y_5 \text{ since } x_5, y_5 \in S_\gamma(b) \cap S_\alpha(a).$$
$$\text{Either } x_6 \overset{\beta}{\to} \!\!\longrightarrow y_{13} \text{ or } x_6 \overset{\delta}{\to} \!\!\longrightarrow y_{13} \text{ since } x_6, y_{13} \in S_\gamma(b) \cap S_\alpha(d).$$
$$\text{Either } x_{15} \overset{\beta}{\to} \!\!\longrightarrow y_9 \text{ or } x_{15} \overset{\delta}{\to} \!\!\longrightarrow y_9 \text{ since } x_{15}, y_9 \in S_\gamma(b) \cap S_\alpha(c).$$

Note further that $x_{15}, x_5, x_2, x_6, x_{14}, y_{15}, y_5, y_3, y_{13}, y_9 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_{15}, x_5, x_2, x_6, x_{14}\} \cap \{y_{15}, y_5, y_3, y_{13}, y_9\} = \emptyset$$
$$P_{\beta,\alpha}(x_{15}, x_5, x_2, x_6, x_{14})$$
$$P_{\beta,\alpha}(y_{15}, y_5, y_3, y_{13}, y_9)$$

Since $\|S_\gamma(b)\| \geq 15$ by (2), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{15}, x_5, x_2, x_6, x_{14})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{15}, y_5, y_3, y_{13}, y_9)$$

to see that if $x_{15} \overset{\delta}{\to} \text{——} y_{15}$, then we get a contradiction. Thus, we cannot have $x_{15} \overset{\delta}{\to} \text{——} y_{15}$, so that, since $x_{15}, y_{15} \in S_\gamma(b) \cap S_\alpha(c)$, we must have $x_{15} \overset{\beta}{\to} \text{——} y_{15}$. But this, combined with (7), implies that

$$N^1_{\delta,\gamma,\beta}(b, x_9, x_{15}, x_{14}, d, y_9, y_{15}, y_{14}).(9)$$

By repeated applications of the symmetry Θ, we see that

$$N^1_{\delta,\beta,\gamma}(c, x_6, x_8, x_{11}, a, y_{13}, y_{10}, y_7), (10)$$

$$N^1_{\delta,\gamma,\beta}(d, x_2, x_5, x_3, b, y_2, y_5, y_3), (11)$$

and

$$N^1_{\delta,\beta,\gamma}(a, x_{13}, x_{10}, x_7, c, y_6, y_8, y_{11}).(12)$$

Finally, we note the following:

$$x_{15} \overset{\delta}{\to} \text{——} y_9 \text{ by (9).}$$
$$x_2 \overset{\delta}{\to} \text{——} y_3 \text{ by (11).}$$
$$x_{14} \overset{\delta}{\to} \text{——} y_{15} \text{ by (9).}$$
$$x_5 \overset{\beta}{\to} \text{——} y_5 \text{ by (11).}$$
$$x_6 \overset{\beta}{\to} \text{——} y_{13} \text{ by (10).}$$
$$x_{15} \overset{\beta}{\to} \text{——} y_{15} \text{ by (9).}$$

Note further that $x_{15}, x_5, x_2, x_6, x_{14}, y_9, y_{13}, y_3, y_5, y_{15} \in S_\gamma(b)$. Note also that the following hold:

$$\{x_{15}, x_5, x_2, x_6, x_{14}\} \cap \{y_9, y_{13}, y_3, y_5, y_{15}\} = \emptyset$$
$$P_{\beta,\alpha}(x_{15}, x_5, x_2, x_6, x_{14})$$
$$P_{\beta,\alpha}(y_9, y_{13}, y_3, y_5, y_{15})$$

Since $\|S_\gamma(b)\| \geq 15$ by (2), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{15}, x_5, x_2, x_6, x_{14})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_9, y_{13}, y_3, y_5, y_{15})$$

to get the desired contradiction. The proof is complete. $\square$

Proposition 31. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \dots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{12} = y_{12}$, $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, and $d = x_{15} = y_{14}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_8, x_2, x_{10}, x_{11}\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{13}, y_8, y_2, y_{10}\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_8, x_2, x_{10}, x_{11})$ and $P_{\alpha,\gamma}(b, y_{13}, y_8, y_2, y_{10})$ hold and that

$$\|\{b, x_8, x_2, x_{10}, x_{11}\} \cap \{b, y_{13}, y_8, y_2, y_{10}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \le 14$ so that

$$\|S_\alpha(a)\|, \|S_\gamma(a)\| \ge 15.(1)$$

Note also that $S_\delta(u) \cap S_\alpha(b) = \{d, x_{11}, x_1, x_8, x_5\} \subseteq S_\alpha(b)$ and $S_\delta(v) \cap S_\alpha(b) = \{d, y_{13}, y_3, y_{10}, y_4\} \subseteq S_\alpha(b)$. Note further that both $P_{\beta,\gamma}(d, x_{11}, x_1, x_8, x_5)$ and $P_{\beta,\gamma}(d, y_{13}, y_3, y_{10}, y_4)$ hold and that

$$\|\{d, x_{11}, x_1, x_8, x_5\} \cap \{d, y_{13}, y_3, y_{10}, y_4\}\| = \|\{d\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(b)\| \le 14$ so that

$$\|S_\beta(b)\|, \|S_\gamma(b)\| \ge 15.(2)$$

Note also that $S_\delta(u) \cap S_\alpha(c) = \{a, x_2, x_5, x_7, x_{11}\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_{13}, y_3, y_6, y_2\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(a, x_2, x_5, x_7, x_{11})$ and $P_{\beta,\gamma}(a, y_{13}, y_3, y_6, y_2)$ hold and that

$$\|\{a, x_2, x_5, x_7, x_{11}\} \cap \{a, y_{13}, y_3, y_6, y_2\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \le 14$ so that

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \ge 15.(3)$$

Note also that $S_\delta(u) \cap S_\beta(d) = \{c, x_5, x_9, x_{10}, x_{11}\} \subseteq S_\beta(d)$ and $S_\delta(v) \cap S_\beta(d) = \{c, y_{13}, y_8, y_4, y_6\} \subseteq S_\beta(d)$. Note further that both $P_{\alpha,\gamma}(c, x_5, x_9, x_{10}, x_{11})$ and $P_{\alpha,\gamma}(c, y_{13}, y_8, y_4, y_6)$ hold and that

$$\|\{c, x_5, x_9, x_{10}, x_{11}\} \cap \{c, y_{13}, y_8, y_4, y_6\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(d)\| \le 14$ so that

$$\|S_\alpha(d)\|, \|S_\gamma(d)\| \ge 15.(4)$$

Now, we note the following:

$$x_9 \overset{\delta}{\to} \text{——}\ y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$

$$x_9 \overset{\delta}{\to} \text{——}\ y_4 \text{ since } x_9, y_4 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$

$$\text{Either } x_9 \overset{\alpha}{\to} \text{——}\ y_8 \text{ or } x_9 \overset{\delta}{\to} \text{——}\ y_8 \text{ since } x_9, y_8 \in S_\gamma(c) \cap S_\beta(d).$$

$$\text{Either } x_9 \overset{\alpha}{\to} \text{——}\ y_7 \text{ or } x_9 \overset{\delta}{\to} \text{——}\ y_7 \text{ since } x_9, y_7 \in S_\gamma(c) \cap S_\beta(b).$$

Note further that $x_1, x_9, x_0, x_{10}, x_3, y_1, y_8, y_0, y_7, y_4 \in s_\gamma(c)$. We also note the following:

$$\{x_1, x_9, x_0, x_{10}, x_3\} \cap \{y_1, y_8, y_0, y_7, y_4\} = \emptyset$$
$$P_{\beta,\alpha}(x_1, x_9, x_0, x_{10}, x_3)$$
$$P_{\beta,\alpha}(y_1, y_8, y_0, y_7, y_4)$$

Since $\|S_\gamma(c)\| \geq 15$ by (3), we may apply Lemma 14 with

$$(u_3, u_4, u_0.u_1, u_2) = (x_1, x_9, x_0, x_{10}, x_3)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_8, y_0, y_7, y_4)$$

to see that either $x_9 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_1, y_8, y_0, y_7, y_4$ or else $x_9 \xrightarrow{\alpha} \!\!\!-\!\!\!-\!\!\!- y_8, y_7$. But we cannot have $x_9 \xrightarrow{\alpha} \!\!\!-\!\!\!-\!\!\!- y_8, y_7$ since $y_8 \xrightarrow{\alpha} \!\!\!-\!\!\!-\!\!\!- y_7$, so we must have

$$x_9 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_1, y_8, y_0, y_7, y_4.(5)$$

Next, we note the following:

$$x_9 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_7 \text{ by (5).}$$
$$x_3 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_7 \text{ since } S_\beta(b) \cap S_\alpha(d) \cap S_\gamma(a).$$

Note further that $a, y_1, y_7, y_6, c, x_7, x_3, x_9 \in S_\beta(b)$. Note also that the following hold:

$$\{y_1, y_7, y_6\} \cap \{x_7, x_3, x_9\} = \emptyset$$
$$P_{\alpha,\gamma}(a, y_1, y_7, y_6, c)$$
$$P_{\alpha,\gamma}(c, x_7, x_3, x_9, a)$$

Since $\|S_\beta(b)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_1, y_7, y_6, c, x_7, x_3, x_9)$$

to get

$$N^2_{\delta,\gamma,\alpha}(a, y_1, y_7, y_6, c, x_7, x_3, x_9).(6)$$

Now, we note the following:

$$x_6 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_5 \text{ since } x_6, y_5 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$
$$x_7 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_6 \text{ by (6).}$$
$$x_3 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_7 \text{ by (6).}$$
$$x_3 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_6 \text{ by (6).}$$
$$x_5 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_6 \text{ since } x_5, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$
$$x_7 \xrightarrow{\alpha} \!\!\!-\!\!\!-\!\!\!- y_7 \text{ by (6).}$$
$$\text{Either } x_3 \xrightarrow{\beta} \!\!\!-\!\!\!-\!\!\!- y_5 \text{ or } x_3 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_5 \text{ since } x_3, y_5 \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_6 \xrightarrow{\beta} \!\!\!-\!\!\!-\!\!\!- y_7 \text{ or } x_6 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_7 \text{ since } x_6, y_7 \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_7 \xrightarrow{\beta} \!\!\!-\!\!\!-\!\!\!- y_3 \text{ or } x_7 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_3 \text{ since } x_7, y_3 \in S_\gamma(a) \cap S_\alpha(c).$$
$$\text{Either } x_5 \xrightarrow{\beta} \!\!\!-\!\!\!-\!\!\!- y_3 \text{ or } x_5 \xrightarrow{\delta} \!\!\!-\!\!\!-\!\!\!- y_3 \text{ since } x_5, y_3 \in S_\gamma(a) \cap S_\alpha(c).$$

Note further that $x_6, x_0, x_7, x_5, x_3, y_5, y_3, y_6, y_0, y_7 \in S_\gamma(a)$. Note also that the following hold:

$$\{x_6, x_0, x_7, x_5, x_3\} \cap \{y_5, y_3, y_6, y_0, y_7\} = \emptyset$$
$$P_{\beta,\alpha}(x_6, x_0, x_7, x_5, x_3)$$
$$P_{\beta,\alpha}(y_5, y_3, y_6, y_0, y_7)$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_6, x_0, x_7, x_5, x_3)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_5, y_3, y_6, y_0, y_7)$$

to see that

$$\begin{aligned} &\text{either } M^2_{\beta,\alpha,\delta}(x_5, x_7, x_0, x_6, x_3, y_6, y_0, y_7, y_5, y_3) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_0, x_7, x_5, x_3, x_6, y_6, y_3, y_5, y_7, y_0). \end{aligned} \tag{7}$$

Next, we note the following:

$$x_6 \overset{\delta}{\to} \text{——} y_5 \text{ by (7).}$$
$$x_2 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_2, y_2 \in S_\gamma(b) \cap S_\alpha(c) \cap S_\beta(a).$$
$$x_4 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_4, y_9 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(c).$$
$$x_0 \overset{\alpha}{\to} \text{——} y_5 \text{ by (7).}$$
$$\text{Either } x_6 \overset{\alpha}{\to} \text{——} y_0 \text{ or } x_6 \overset{\delta}{\to} \text{——} y_0 \text{ by (7).}$$
$$\text{Either } x_{10} \overset{\alpha}{\to} \text{——} y_2 \text{ or } x_{10} \overset{\delta}{\to} \text{——} y_2 \text{ since } x_{10}, y_2 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_{10} \overset{\alpha}{\to} \text{——} y_8 \text{ or } x_{10} \overset{\delta}{\to} \text{——} y_8 \text{ since } x_{10}, y_8 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_4 \overset{\alpha}{\to} \text{——} y_5 \text{ or } x_4 \overset{\delta}{\to} \text{——} y_5 \text{ since } x_4, y_5 \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_2 \overset{\alpha}{\to} \text{——} y_8 \text{ or } x_2 \overset{\delta}{\to} \text{——} y_8 \text{ since } x_2, y_8 \in S_\gamma(b) \cap S_\beta(a).$$
$$\text{Either } x_6 \overset{\alpha}{\to} \text{——} y_9 \text{ or } x_6 \overset{\delta}{\to} \text{——} y_9 \text{ since } x_6, y_9 \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_6 \overset{\beta}{\to} \text{——} y_2 \text{ or } x_6 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_6, y_2 \in S_\gamma(b) \cap S_\alpha(d).$$
$$\text{Either } x_2 \overset{\beta}{\to} \text{——} y_5 \text{ or } x_2 \overset{\delta}{\to} \text{——} y_5 \text{ since } x_2, y_5 \in S_\gamma(b) \cap S_\alpha(d).$$

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_8, y_5, y_2, y_9, y_0 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_8, y_5, y_2, y_9, y_0\} = \emptyset$$
$$P_{\beta,\alpha}(x_0, x_6, x_2, x_4, x_{10})$$
$$P_{\beta,\alpha}(y_8, y_5, y_2, y_9, y_0)$$

Since $\|S_\gamma(b)\| \geq 15$ by (2), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_8, y_5, y_2, y_9, y_0)$$

to see that

$$\begin{aligned} &\text{either } M^1_{\beta,\alpha,\delta}(x_6, x_0, x_{10}, x_4, x_2, y_5, y_8, y_0, y_9, y_2) \\ &\quad\text{or } M^0_{\beta,\alpha,\delta}(x_{10}, x_0, x_6, x_2, x_4, y_8, y_0, y_9, y_2, y_5) \\ &\quad\text{or } M^1_{\beta,\alpha,\delta}(x_6, x_0, x_{10}, x_4, x_2, y_5, y_2, y_9, y_0, y_8). \end{aligned} \tag{8}$$

Note that $x_2 \overset{\beta}{\to} \!\!—\!\!—\, y_5$ by (8). We cannot have $x_5 \overset{\beta}{\to} \!\!—\!\!—\, y_5$, since if we did, the fact that $x_2 \overset{\beta}{\to} \!\!—\!\!—\, x_5$ would result in a monochromatic triangle. Since we cannot have $x_5 \overset{\beta}{\to} \!\!—\!\!—\, y_5$, we see from (7) that

$$M^2_{\beta,\alpha,\delta}(x_0, x_7, x_5, x_3, x_6, y_6, y_3, y_5, y_7, y_0).(9)$$

Now, note that $x_6 \overset{\delta}{\to} \!\!—\!\!—\, y_0$ by (9), so that we see from (8) that

$$M^0_{\beta,\alpha,\delta}(x_{10}, x_0, x_6, x_2, x_4, y_8, y_0, y_9, y_2, y_5).(1)0$$

By (9), we have $x_3 \overset{\beta}{\to} \!\!—\!\!—\, y_0$. By (10), we have $x_{10} \overset{\beta}{\to} \!\!—\!\!—\, y_0$. But this gives a monochromatic triangle since $x_3 \overset{\beta}{\to} \!\!—\!\!—\, x_{10}$, thus producing the desired contradiction. The proof is complete. □

Proposition 32. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, e\}$ where $a = x_{12} = y_{12}$, $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, and $e = x_{11} = y_{13}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\alpha(a) = \{c, x_{15}, x_9, x_1, x_4\} \subseteq S_\alpha(a)$ and $S_\delta(v) \cap S_\alpha(a) = \{c, y_9, y_1, y_4, y_{14}\} \subseteq S_\alpha(a)$. Note further that both $P_{\beta,\gamma}(c, x_{15}, x_9, x_1, x_4)$ and $P_{\beta,\gamma}(c, y_9, y_1, y_4, y_{14})$ hold and that

$$\|\{c, x_{15}, x_9, x_1, x_4\} \cap \{c, y_9, y_1, y_4, y_{14}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(a)\| \le 14$ so that

$$\|S_\beta(a)\|, \|S_\gamma(a)\| \ge 15.(1)$$

Note also that $S_\delta(u) \cap S_\alpha(b) = \{e, x_{15}, x_5, x_8, x_1\} \subseteq S_\alpha(b)$ and $S_\delta(v) \cap S_\alpha(b) = \{e, y_3, y_{10}, y_4, y_{14}\} \subseteq S_\alpha(b)$. Note further that both $P_{\beta,\gamma}(e, x_{15}, x_5, x_8, x_1)$ and $P_{\beta,\gamma}(e, y_3, y_{10}, y_4, y_{14})$ hold and that

$$\|\{e, x_{15}, x_5, x_8, x_1\} \cap \{e, y_3, y_{10}, y_4, y_{14}\}\| = \|\{e\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(b)\| \le 14$ so that

$$\|S_\beta(b)\|, \|S_\gamma(b)\| \ge 15.(2)$$

Note also that $S_\delta(u) \cap S_\beta(c) = \{b, x_8, x_4, x_6, x_{15}\} \subseteq S_\beta(c)$ and $S_\delta(v) \cap S_\beta(c) = \{b, y_{10}, y_9, y_5, y_{14}\} \subseteq S_\beta(c)$. Note further that both $P_{\alpha,\gamma}(b, x_8, x_4, x_6, x_{15})$ and $P_{\alpha,\gamma}(b, y_{10}, y_9, y_5, y_{14})$ hold and that

$$\|\{b, x_8, x_4, x_6, x_{15}\} \cap \{b, y_{10}, y_9, y_5, y_{14}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(c)\| \le 14$ so that

$$\|S_\alpha(c)\|, \|S_\gamma(c)\| \ge 15.(3)$$

Note also that $S_\delta(u) \cap S_\beta(e) = \{a, x_1, x_7, x_6, x_{15}\} \subseteq S_\beta(e)$ and $S_\delta(v) \cap S_\beta(e) = \{a, y_9, y_3, y_7, y_{14}\} \subseteq S_\beta(e)$. Note further that both $P_{\alpha,\gamma}(a, x_1, x_7, x_6, x_{15})$ and $P_{\alpha,\gamma}(a, y_9, y_3, y_7, y_{14})$ hold and that

$$\|\{a, x_1, x_7, x_6, x_{15}\} \cap \{a, y_9, y_3, y_7, y_{14}\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(e)\| \leq 14$ so that

$$\|S_\alpha(e)\|, \|S_\gamma(e)\| \geq 15. (4)$$

Note that $x_2 \xrightarrow{\delta} y_2$ since $x_2, y_2 \in S_\beta(a) \cap S_\alpha(c) \cap S_\gamma(e)$. Note further that $e, y_8, y_2, y_{10}, b, x_8, x_2, x_{10} \in S_\beta(a)$. Note also that the following hold:

$$\{y_8, y_2, y_{10}\} \cap \{x_8, x_2, x_{10}\} = \emptyset$$
$$P_{\alpha,\gamma}(e, y_8, y_2, y_{10}, b)$$
$$P_{\alpha,\gamma}(b, x_8, x_2, x_{10}, e)$$

Since $\|S_\beta(a)\| \geq 15$ by (1), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (e, y_8, y_2, y_{10}, b, x_8, x_2, x_{10})$$

to see that

$$\begin{aligned} &\text{either } N^2_{\delta,\gamma,\alpha}(e, y_8, y_2, y_{10}, b, x_8, x_2, x_{10}) \\ &\text{or } N^2_{\delta,\gamma,\alpha}(e, x_{10}, x_2, x_8, b, y_{10}, y_2, y_8). \end{aligned} \tag{5}$$

Next, we note the following:

$$x_8 \xrightarrow{\delta} y_{10} \text{ since } x_8, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(a).$$
$$x_2 \xrightarrow{\delta} y_2 \text{ since } x_2, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(a).$$
$$\text{Either } x_5 \xrightarrow{\beta} y_2 \text{ or } x_5 \xrightarrow{\delta} y_2 \text{ since } x_5, y_2 \in S_\gamma(e) \cap S_\alpha(c).$$
$$\text{Either } x_5 \xrightarrow{\beta} y_4 \text{ or } x_5 \xrightarrow{\delta} y_4 \text{ since } x_5, y_4 \in S_\gamma(e) \cap S_\alpha(b).$$
$$\text{Either } x_2 \xrightarrow{\beta} y_6 \text{ or } x_2 \xrightarrow{\delta} y_6 \text{ since } x_2, y_6 \in S_\gamma(e) \cap S_\alpha(c).$$
$$\text{Either } x_5 \xrightarrow{\beta} y_{10} \text{ or } x_5 \xrightarrow{\delta} y_{10} \text{ since } x_5, y_{10} \in S_\gamma(e) \cap S_\alpha(b).$$
$$\text{Either } x_8 \xrightarrow{\beta} y_4 \text{ or } x_8 \xrightarrow{\delta} y_4 \text{ since } x_8, y_4 \in S_\gamma(e) \cap S_\alpha(b).$$
$$\text{Either } x_5 \xrightarrow{\beta} y_6 \text{ or } x_5 \xrightarrow{\delta} y_6 \text{ since } x_5, y_6 \in S_\gamma(e) \cap S_\alpha(b).$$
$$\text{Either } x_9 \xrightarrow{\beta} y_4 \text{ or } x_9 \xrightarrow{\delta} y_4 \text{ since } x_9, y_4 \in S_\gamma(e) \cap S_\alpha(a).$$
$$\text{Either } x_8 \xrightarrow{\alpha} y_2 \text{ or } x_8 \xrightarrow{\delta} y_2 \text{ since } x_8, y_2 \in S_\gamma(e) \cap S_\beta(a).$$
$$\text{Either } x_2 \xrightarrow{\alpha} y_{10} \text{ or } x_2 \xrightarrow{\delta} y_{10} \text{ since } x_2, y_{10} \in S_\gamma(e) \cap S_\beta(a).$$

Note further that $x_9, x_0, x_8, x_5, x_2, y_6, y_0, y_{10}, y_4, y_2 \in S_\gamma(e)$. Note also that the following hold:

$$\{x_9, x_0, x_8, x_5, x_2\} \cap \{y_6, y_0, y_{10}, y_4, y_2\} = \emptyset$$
$$P_{\beta,\alpha}(x_9, x_0, x_8, x_5, x_2)$$
$$P_{\beta,\alpha}(y_6, y_0, y_{10}, y_4, y_2)$$

Since $\|S_\gamma(e)\| \geq 15$ by (4), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_8, x_5, x_2)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_6, y_0, y_{10}, y_4, y_2)$$

to see that

$$\text{if } x_9 \overset{\delta}{\to} \text{——} y_6, \text{ then } M^0_{\beta,\alpha,\gamma}(x_8, x_5, x_2, x_9, x_0, y_{10}, y_4, y_2, y_6, y_0).(6)$$

Now, if $x_9 \overset{\delta}{\to} \text{——} y_6$, then we would have both $x_8 \overset{\delta}{\to} \text{——} y_2$ and $x_2 \overset{\delta}{\to} \text{——} y_{10}$ by (6), which would then contradict (5). Thus, we cannot have $x_9 \overset{\delta}{\to} \text{——} y_6$, so that, since $x_9, y_6 \in S_\gamma(e) \cap S_\beta(b)$, we must have

$$x_9 \overset{\alpha}{\to} \text{——} y_6.(7)$$

Note that $x_9, x_0, x_8, x_5, x_2, y_6, y_0, y_{10}, y_4, y_2 \in S_\gamma(e)$. Note also that the following hold:

$$\{x_9, x_0, x_8, x_5, x_2\} \cap \{y_6, y_0, y_{10}, y_4, y_2\} = \emptyset$$
$$P_{\beta,\alpha}(x_9, x_0, x_8, x_5, x_2)$$
$$P_{\beta,\alpha}(y_6, y_0, y_{10}, y_4, y_2)$$

Noting also that $\|S_\gamma(e)\| \geq 15$ by (4), we will apply Lemma 14 several times in what follows with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_8, x_5, x_2)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_6, y_0, y_{10}, y_4, y_2).$$

We now show that $x_5 \overset{\delta}{\to} \text{——} y_4$. Suppose not. Then $x_5 \overset{\beta}{\to} \text{——} y_4$ since $x_5, y_4 \in S_\gamma(e) \cap S_\alpha(b)$. We also note the following:

$$x_9 \overset{\alpha}{\to} \text{——} y_6 \text{ by (7).}$$
$$x_8 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_8, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(a).$$
$$x_2 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_2, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(a).$$
$$x_5 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_5, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(y_4).$$
$$x_5 \overset{\delta}{\to} \text{——} y_{10} \text{ since } x_5, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(y_4).$$
$$x_5 \overset{\beta}{\to} \text{——} y_6 \text{ by Lemma 14, since } x_5 \overset{\delta}{\to} \text{——} y_2, y_{10} \text{ and } x_5 \overset{\beta}{\to} \text{——} y_4 \text{ and } x_5, y_6 \in S_\alpha(x_9).$$
$$x_5 \overset{\alpha}{\to} \text{——} y_6 \text{ by Lemma 14, since } x_5 \overset{\delta}{\to} \text{——} y_2, y_{10} \text{ and } x_5 \overset{\beta}{\to} \text{——} y_4, y_6.$$
$$x_2 \overset{\delta}{\to} \text{——} y_6 \text{ since } x_2, y_6 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(x_5).$$
$$x_8 \overset{\delta}{\to} \text{——} y_4 \text{ since } x_8, y_4 \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(x_5).$$
$$x_8 \overset{\delta}{\to} \text{——} y_6 \text{ since } x_8, y_6 \in S_\gamma(e) \cap S_\alpha(x_9) \cap S_\beta(x_5).$$
$$x_8 \overset{\delta}{\to} \text{——} y_2, y_0 \text{ by Lemma 14, since } x_8 \overset{\delta}{\to} \text{——} y_6, y_{10}, y_4.$$

Now, by Lemma 14, since $x_9 \overset{\alpha}{\to} \text{——} x_8 \overset{\delta}{\to} \text{——} y_6, y_0, y_{10}, y_4, y_2$, we must have two α-colored edges from x_9 to $\{y_6, y_0, y_{10}, y_4, y_2\}$. But $x_9 \overset{\delta}{\to} \text{——} y_2$ and $x_9, y_4, y_{10} \in S_\alpha(y_6)$ and $x_9, x_0 \in S_\alpha(y_5)$, so that this is impossible, thus giving a contradiction. Therefore, we have

$$x_5 \overset{\delta}{\to} \text{——} y_4, (8)$$

as desired.

Finally, we note the following:

$$x_8 \overset{\delta}{\rightarrow} \text{——} \, y_{10} \text{ since } x_8, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(a).$$
$$x_5 \overset{\delta}{\rightarrow} \text{——} \, y_4 \text{ by } (8).$$
$$x_2 \overset{\delta}{\rightarrow} \text{——} \, y_2 \text{ since } x_2, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(a).$$
$$x_9 \overset{\alpha}{\rightarrow} \text{——} \, y_6 \text{ by } (7).$$
$$\text{Either } x_2 \overset{\beta}{\rightarrow} \text{——} \, y_6 \text{ or } x_2 \overset{\delta}{\rightarrow} \text{——} \, y_6 \text{ since } x_2, y_6 \in S_\gamma(e) \cap S_\alpha(c).$$

Note further that $x_0, x_8, x_5, x_2, x_9, y_0, y_{10}, y_4, y_2, y_6 \in S_\gamma(e)$. Note also that the following hold:

$$\{x_0, x_8, x_5, x_2, x_9\} \cap \{y_0, y_{10}, y_4, y_2, y_6\} = \emptyset$$
$$P_{\beta,\alpha}(x_0, x_8, x_5, x_2, x_9)$$
$$P_{\beta,\alpha}(y_0, y_{10}, y_4, y_2, y_6)$$

Since $\|S_\gamma(e)\| \geq 15$ by (4), we may apply Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_8, x_5, x_2, x_9)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_{10}, y_4, y_2, y_6)$$

to see that

$$M^1_{\beta,\alpha,\delta}(x_2, x_9, x_0, x_8, x_5, y_2, y_4, y_{10}, y_0, y_6).(9)$$

Finally, we note the following:

$$x_5 \overset{\beta}{\rightarrow} \text{——} \, y_2 \text{ by } (9).$$
$$x_7 \overset{\delta}{\rightarrow} \text{——} \, y_6 \text{ since } x_7, y_6 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(b).$$

Note further that $a, y_2, y_6, y_3, e, x_7, x_5, x_2 \in S_\alpha(c)$. Note also that the following hold:

$$\{y_2, y_6, y_3\} \cap \{x_7, x_5, x_2\} = \emptyset$$
$$P_{\beta,\gamma}(a, y_2, y_6, y_3, e)$$
$$P_{\beta,\gamma}(e, x_7, x_5, x_2, a)$$

Since $\|S_\alpha(c)\| \geq 15$ by (3), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_2, y_6, y_3, e, x_7, x_5, x_2)$$

to produce the desired contradiction. The theorem is proved. $\square$

Proposition 33. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ where $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, $d = x_{15} = y_{14}$, and $e = x_{11} = y_{13}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto v \longmapsto u$$
$$b \longmapsto e \longmapsto b$$
$$c \longmapsto d \longmapsto c$$
$$x_i \longmapsto y_i \longmapsto x_i \quad \text{for all} \quad i \in \{0, \dots, 15\}$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \alpha$$
$$\beta \longmapsto \beta$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\alpha(c) = \{e, x_{12}, x_2, x_5, x_7\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{e, y_{12}, y_2, y_6, y_3\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(e, x_{12}, x_2, x_5, x_7)$ and $P_{\beta,\gamma}(e, y_{12}, y_2, y_6, y_3)$ hold and that

$$\|\{e, x_{12}, x_2, x_5, x_7\} \cap \{e, y_{12}, y_2, y_6, y_3\}\| = \|\{e\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15. (1)$$

An application of the symmetry Θ to (1) gives

$$\|S_\beta(d)\|, \|S_\gamma(d)\| \geq 15. (2)$$

First, note that $c, y_6, y_4, y_8, e, x_{10}, x_9, x_5 \in S_\beta(d)$. Note also that the following hold:

$$\{y_6, y_4, y_8\} \cap \{x_{10}, x_9, x_5\} = \emptyset$$
$$P_{\alpha,\gamma}(c, y_6, y_4, y_8, e)$$
$$P_{\alpha,\gamma}(e, x_{10}, x_9, x_5, c)$$

Since $\|S_\beta(d)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (c, y_6, y_4, y_8, e, x_{10}, x_9, x_5)$$

to see that

$$\text{if } x_9 \overset{\alpha}{\to} \text{——} \; y_8, \text{ then } N^2_{\delta,\gamma,\alpha}(c, y_6, y_4, y_8, e, x_{10}, x_9, x_5). (3)$$

Next, we show that $x_9 \overset{\delta}{\to} \text{——}\; y_8$. Suppose not. Then $x_9 \overset{\alpha}{\to} \text{——}\; y_8$, since $x_9, y_8 \in S_\beta(d) \cap S_\gamma(c)$. But then, we have $x_5 \overset{\alpha}{\to} \text{——}\; y_4$ by (3), which is impossible since $x_5, y_4 \in S_\alpha(b)$. Thus,

$$x_9 \overset{\delta}{\to} \text{——}\; y_8, (4)$$

as desired.

Note that $x_3, x_1, x_9, x_0, x_{10}, y_7, y_0, y_8, y_1, y_4 \in S_\gamma(c)$. Note also that the following hold:

$$\{x_3, x_1, x_9, x_0, x_{10}\} \cap \{y_7, y_0, y_8, y_1, y_4\} = \emptyset$$
$$P_{\beta,\alpha}(x_3, x_1, x_9, x_0, x_{10})$$
$$P_{\beta,\alpha}(y_7, y_0, y_8, y_1, y_4)$$

Noting also that $\|S_\gamma(e)\| \geq 15$ by (1), we will apply Lemma 14 several times in what follows with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_7, y_0, y_8, y_1, y_4).$$

We now show that $x_{10} \overset{\delta}{\to} \text{——}\, y_4$. Suppose not. Then $x_{10} \overset{\delta}{\to} \text{——}\, y_4$, since $x_{10}, y_4 \in S_\gamma(c) \cap S_\beta(d)$. We also note the following:

$$x_9 \overset{\delta}{\to} \text{——}\, y_4 \text{ since } x_9, y_4 \in S_\gamma(c) \cap S_\alpha(x_{10}) \cap S_\beta(d).$$

$$\text{Either } x_9 \overset{\alpha}{\to} \text{——}\, y_1 \text{ or } x_9 \overset{\delta}{\to} \text{——}\, y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\beta(b).$$

$$\text{Either } x_9 \overset{\alpha}{\to} \text{——}\, y_7 \text{ or } x_9 \overset{\delta}{\to} \text{——}\, y_7 \text{ since } x_9, y_7 \in S_\gamma(c) \cap S_\beta(b).$$

Also, $x_9 \overset{\delta}{\to} \text{——}\, y_1, y_0, y_7$ since otherwise we would have, by Lemma 14, in view of the fact that $x_9 \overset{\delta}{\to} \text{——}\, y_8, y_4$, that $x_9 \overset{\alpha}{\to} \text{——}\, y_1, y_7$, which is impossible since $y_1 \overset{\alpha}{\to} \text{——}\, y_7$. But now, since $x_3 \overset{\delta}{\to} \text{——}\, y_1, y_7$ and $x_3 \overset{\alpha}{\to} \text{——}\, y_9$, we must have two α-colored edges from x_3 to $\{y_8, y_0, y_4\}$, by Lemma 14. But $x_3, y_8 \in S_\alpha(e)$, so that we must have $x_3 \overset{\alpha}{\to} \text{——}\, y_0, y_4$. but this gives a contradiction since $y_0 \overset{\alpha}{\to} \text{——}\, y_4$. Thus, we have

$$x_{10} \overset{\delta}{\to} \text{——}\, y_4, (5)$$

as desired.

Now, note that the following hold:

$$x_3 \overset{\delta}{\to} \text{——}\, y_7 \text{ since } x_3, y_7 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).$$

$$x_9 \overset{\delta}{\to} \text{——}\, y_8 \text{ by (4).}$$

$$x_{10} \overset{\delta}{\to} \text{——}\, y_4 \text{ by (5).}$$

$$x_3 \overset{\delta}{\to} \text{——}\, y_1 \text{ since } x_3, y_1 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(b).$$

$$x_{10} \overset{\delta}{\to} \text{——}\, y_8 \text{ since } x_{10}, y_8 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(d).$$

$$\text{Either } x_1 \overset{\alpha}{\to} \text{——}\, y_7 \text{ or } x_1 \overset{\delta}{\to} \text{——}\, y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\beta(e).$$

$$\text{Either } x_3 \overset{\beta}{\to} \text{——}\, y_8 \text{ or } x_3 \overset{\delta}{\to} \text{——}\, y_8 \text{ since } x_3, y_8 \in S_\gamma(c) \cap S_\alpha(e).$$

$$\text{Either } x_9 \overset{\alpha}{\to} \text{——}\, y_1 \text{ or } x_9 \overset{\delta}{\to} \text{——}\, y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\beta(b).$$

Note further that $x_3, x_1, x_9, x_0, x_{10}, y_7, y_0, y_8, y_1, y_4 \in S_\gamma(c)$. Note also that the following hold:

$$\{x_3, x_1, x_9, x_0, x_{10}\} \cap \{y_7, y_0, y_8, y_1, y_4\} = \emptyset$$
$$P_{\beta,\alpha}(x_3, x_1, x_9, x_0, x_{10})$$
$$P_{\beta,\alpha}(y_7, y_0, y_8, y_1, y_4)$$

Since $\|S_\gamma(e)\| \geq 15$ by (1), we ma apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_7, y_0, y_8, y_1, y_4)$$

to get the desired contradiction. The proof is complete. □

Proposition 34. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such* *that* $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ *where* $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, *and* $e = x_{12} = y_{12}$. *Then the graph has fewer than* 62 *vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_{13} \longmapsto x_{13}$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_7 \longmapsto x_5 \longmapsto x_9 \longmapsto x_1$$
$$x_2 \longmapsto x_{10} \longmapsto x_4 \longmapsto x_6 \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_{13} \longmapsto y_{13}$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1$$
$$y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(b) = \{c, x_6, x_4, x_8, x_{13}\} \subseteq S_\beta(b)$ and $S_\delta(v) \cap S_\beta(b) = \{c, y_6, y_4, y_8, y_{13}\} \subseteq S_\beta(b)$. Note further that both $P_{\alpha,\gamma}(c, x_6, x_4, x_8, x_{13})$ and $P_{\alpha,\gamma}(c, y_6, y_4, y_8, y_{13})$ hold and that

$$\|\{c, x_6, x_4, x_8, x_{13}\} \cap \{c, y_6, y_4, y_8, y_{13}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(b)\| \leq 14$ so that

$$\|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15.(1)$$

An application of the symetry Θ to (1) gives

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.(2)$$

Now, note that $x_9 \overset{\delta}{\to} \text{——}\ y_9$. Note further that $d, y_{10}, y_9, y_5, b, x_5, x_9, x_{10} \in S_\beta(c)$. Note also that the following hold:

$$\{y_{10}, y_9, y_5\} \cap \{x_5, x_9, x_{10}\} = \emptyset$$
$$P_{\alpha,\gamma}(d, y_{10}, y_9, y_5, b)$$
$$P_{\alpha,\gamma}(b, x_5, x_9, x_{10}, d)$$

Since $\|S_\beta(c)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (d, y_{10}, y_9, y_5, b, x_5, x_9, x_{10})$$

to see that

$$\text{either } N^2_{\delta,\gamma,\alpha}(d, y_{10}, y_9, y_5, b, x_5, x_9, x_{10})$$

(3) $$\text{or } N^2_{\delta,\gamma,\alpha}(d, x_{10}, x_9, x_5, b, y_5, y_9, y_{10}).$$

Next, note that either $x_{10} \xrightarrow{\delta}$ —— y_9 or $x_9 \xrightarrow{\delta}$ —— y_{10} by (3). Note also that the following hold:

$$x_9 \xrightarrow{\delta} \text{——}\ y_9 \text{ since } x_9, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(c).$$

$$x_{10} \xrightarrow{\delta} \text{——}\ y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c).$$

$$x_1 \xrightarrow{\delta} \text{——}\ y_1 \text{ since } x_1, y_1 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(d).$$

$$\text{Either } x_3 \xrightarrow{\beta} \text{——}\ y_3 \text{ or } x_3 \xrightarrow{\delta} \text{——}\ y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(c).$$

$$\text{Either } x_3 \xrightarrow{\beta} \text{——}\ y_{10} \text{ or } x_3 \xrightarrow{\delta} \text{——}\ y_{10} \text{ since } x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_{10} \xrightarrow{\beta} \text{——}\ y_3 \text{ or } x_{10} \xrightarrow{\delta} \text{——}\ y_3 \text{ since } x_{10}, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_1 \xrightarrow{\beta} \text{——}\ y_9 \text{ or } x_1 \xrightarrow{\delta} \text{——}\ y_9 \text{ since } x_1, y_9 \in S_\gamma(b) \cap S_\alpha(e).$$

$$\text{Either } x_9 \xrightarrow{\beta} \text{——}\ y_1 \text{ or } x_9 \xrightarrow{\delta} \text{——}\ y_1 \text{ since } x_9, y_1 \in S_\gamma(b) \cap S_\alpha(e).$$

$$\text{Either } x_9 \xrightarrow{\alpha} \text{——}\ y_{10} \text{ or } x_9 \xrightarrow{\delta} \text{——}\ y_{10} \text{ since } x_9, y_{10} \in S_\gamma(b) \cap S_\beta(c).$$

$$\text{Either } x_{10} \xrightarrow{\alpha} \text{——}\ y_9 \text{ or } x_{10} \xrightarrow{\delta} \text{——}\ y_9 \text{ since } x_{10}, y_9 \in S_\gamma(b) \cap S_\beta(c).$$

Note further that $x_9, x_0, x_{10}, x_3, x_1, y_9, y_0, y_{10}, y_3, y_1 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_9, x_0, x_{10}, x_3, x_1\} \cap \{y_9, y_0, y_{10}, y_3, y_1\} = \emptyset$$

$$P_{\beta,\alpha}(x_9, x_0, x_{10}, x_3, x_1)$$

$$P_{\beta,\alpha}(y_9, y_0, y_{10}, y_3, y_1)$$

Since $\|S_\gamma(b)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_{10}, x_3, x_1)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_9, y_0, y_{10}, y_3, y_1)$$

to see that

$$\text{either } M^2_{\beta,\alpha,\delta}(x_3, x_{10}, x_0, x_9, x_1, y_{10}, y_3, y_1, y_9, y_0)$$

(4) $$\text{or } M^2_{\beta,\alpha,\delta}(x_0, x_9, x_1, x_3, x_{10}, y_1, y_9, y_0, y_{10}, y_3).$$

By (4), we have

$$x_0 \xrightarrow{\alpha} \text{——}\ y_0. (5)$$

Applying the symmetry Θ to (5) gives

$$x_0 \xrightarrow{\beta} \text{——}\ y_0. (6)$$

(5) and (6) together produce the desired contradiction. The proof is complete. ☐

Proposition 35. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_{11} = y_{11}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u)\cap S_\beta(a) = \{b, x_7, x_3, x_9, x_{12}\} \subseteq S_\beta(a)$ and $S_\delta(v)\cap S_\beta(a) = \{b, y_7, y_3, y_9, y_{12}\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_7, x_3, x_9, x_{12})$ and $P_{\alpha,\gamma}(b, y_7, y_3, y_9, y_{12})$ hold and that

$$\|\{b, x_7, x_3, x_9, x_{12}\} \cap \{b, y_7, y_3, y_9, y_{12}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$\|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.(1)$$

Note also that $S_\delta(u) \cap S_\alpha(c) = \{a, x_{12}, x_2, x_6, x_3\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_{12}, y_2, y_6, y_3\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(a, x_{12}, x_2, x_6, x_3)$ and $P_{\beta,\gamma}(a, y_{12}, y_2, y_6, y_3)$ hold and that

$$\|\{a, x_{12}, x_2, x_6, x_3\} \cap \{a, y_{12}, y_2, y_6, y_3\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.(2)$$

Now, note that $x_8 \overset{\delta}{\to} \text{——}\ y_8$ since $x_8, y_8 \in S_\alpha(a)\cap S_\beta(b)\cap S_\gamma(c)$. Note further that $d, y_1, y_8, y_5, c, x_5, x_8, x_1 \in S_\alpha(a)$. Note also that the following hold:

$$\{y_1, y_8, y_5\} \cap \{x_5, x_8, x_1\} = \emptyset$$
$$P_{\beta,\alpha}(d, y_1, y_8, y_5, c)$$
$$P_{\beta,\alpha}(c, x_5, x_8, x_1, d)$$

Since $\|S_\alpha(a)\| \geq 15$ by (1), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (d, y_1, y_8, y_5, c, x_5, x_8, x_1)$$

to see that

$$\text{either } N^2_{\delta,\gamma,\beta}(d, y_1, y_8, y_5, c, x_5, x_8, x_1)$$
$$\text{or } N^2_{\delta,\gamma,\beta}(d, x_1, x_8, x_5, c, y_5, y_8, y_1). \tag{3}$$

Thus, by (3), we see that

$$\text{either } x_8 \overset{\beta}{\to} \text{——}\ y_1 \text{ or } x_1 \overset{\beta}{\to} \text{——}\ y_8.(4)$$

Also, by (3), we see that

$$\text{either } x_1 \overset{\delta}{\to} \text{——}\ y_8 \text{ or } x_8 \overset{\delta}{\to} \text{——}\ y_1.(5)$$

Next, we note the following:

$$x_1 \overset{\delta}{\to} \text{——}\ y_1 \text{ since } x_1, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$
$$x_8 \overset{\delta}{\to} \text{——}\ y_8 \text{ since } x_8, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$
$$x_7 \overset{\delta}{\to} \text{——}\ y_7 \text{ since } x_7, y_7 \in S_\gamma(c) \cap S_\alpha(b) \cap S_\beta(a).$$
$$x_4 \overset{\delta}{\to} \text{——}\ y_4 \text{ since } x_4, y_4 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).$$
$$\text{Either } x_1 \overset{\alpha}{\to} \text{——}\ y_7 \text{ or } x_1 \overset{\delta}{\to} \text{——}\ y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\beta(d).$$
$$\text{Either } x_8 \overset{\alpha}{\to} \text{——}\ y_4 \text{ or } x_8 \overset{\delta}{\to} \text{——}\ y_4 \text{ since } x_8, y_4 \in S_\gamma(c) \cap S_\beta(b).$$
$$\text{Either } x_7 \overset{\alpha}{\to} \text{——}\ y_1 \text{ or } x_7 \overset{\delta}{\to} \text{——}\ y_1 \text{ since } x_7, y_1 \in S_\gamma(c) \cap S_\beta(d).$$
$$\text{Either } x_4 \overset{\alpha}{\to} \text{——}\ y_8 \text{ or } x_4 \overset{\delta}{\to} \text{——}\ y_8 \text{ since } x_4, y_8 \in S_\gamma(c) \cap S_\beta(b).$$

Note further that $x_1, x_8, x_0, x_7, x_4, y_1, y_8, y_0, y_7, y_4 \in S_\gamma(c)$. Note also that the following hold:

$$\begin{gathered}\{x_1, x_8, x_0, x_7, x_4\} \cap \{y_1, y_8, y_0, y_7, y_4\} = \emptyset \\ P_{\beta,\alpha}(x_1, x_8, x_0, x_7, x_4) \\ P_{\beta,\alpha}(y_1, y_8, y_0, y_7, y_4)\end{gathered}$$

Since $\|S_\gamma(c)\| \geq 15$ by (2), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_1, x_8, x_0, x_7, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_8, y_0, y_7, y_4)$$

to see, with the help of (4) and (5), that

$$\begin{gathered}\text{either } M^2_{\beta,\alpha,\gamma}(x_0, x_7, x_4, x_1, x_8, y_0, y_7, y_4, y_1, y_8) \\ \text{or } M^2_{\beta,\alpha,\gamma}(x_8, x_1, x_4, x_7, x_0, y_8, y_1, y_4, y_7, y_0).\end{gathered} \tag{6}$$

Thus, we have

$$\text{either } x_0 \stackrel{\alpha}{\to} \text{——}\, y_4 \text{ or } x_4 \stackrel{\alpha}{\to} \text{——}\, y_0, (7)$$

by (6).

Finally, we note the following:

$$\begin{gathered}
x_{10} \stackrel{\delta}{\to} \text{——}\, y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c). \\
x_4 \stackrel{\delta}{\to} \text{——}\, y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b). \\
x_6 \stackrel{\delta}{\to} \text{——}\, y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b). \\
x_0 \stackrel{\delta}{\to} \text{——}\, y_0 \text{ by (6).} \\
\text{Either } x_{10} \stackrel{\beta}{\to} \text{——}\, y_4 \text{ or } x_{10} \stackrel{\delta}{\to} \text{——}\, y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\alpha(d). \\
\text{Either } x_4 \stackrel{\beta}{\to} \text{——}\, y_{10} \text{ or } x_4 \stackrel{\delta}{\to} \text{——}\, y_{10} \text{ since } x_4, y_{10} \in S_\gamma(a) \cap S_\alpha(d). \\
\text{Either } x_2 \stackrel{\beta}{\to} \text{——}\, y_6 \text{ or } x_2 \stackrel{\delta}{\to} \text{——}\, y_6 \text{ since } x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c). \\
\text{Either } x_6 \stackrel{\beta}{\to} \text{——}\, y_2 \text{ or } x_6 \stackrel{\delta}{\to} \text{——}\, y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c). \\
\text{Either } x_6 \stackrel{\alpha}{\to} \text{——}\, y_4 \text{ or } x_6 \stackrel{\delta}{\to} \text{——}\, y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b). \\
\text{Either } x_4 \stackrel{\alpha}{\to} \text{——}\, y_6 \text{ or } x_4 \stackrel{\delta}{\to} \text{——}\, y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\beta(b).
\end{gathered}$$

Note further that $x_{10}, x_4, x_2, x_6, x_0, y_{10}, y_4, y_2, y_6, y_0 \in S_\gamma(a)$. Note also that the following hold:

$$\begin{gathered}\{x_{10}, x_4, x_2, x_6, x_0\} \cap \{y_{10}, y_4, y_2, y_6, y_0\} = \emptyset \\ P_{\beta,\alpha}(x_{10}, x_4, x_2, x_6, x_0) \\ P_{\beta,\alpha}(y_{10}, y_4, y_2, y_6, y_0)\end{gathered}$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{10}, x_4, x_2, x_6, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{10}, y_4, y_2, y_6, y_0)$$

to get, with the help of (7), the desired contradiction. The proof is complete. $\square$

Proposition 36. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^0_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ where $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_{13} \longmapsto x_{13}$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_9 \longmapsto x_5 \longmapsto x_7 \longmapsto x_1$$
$$x_2 \longmapsto x_6 \longmapsto x_4 \longmapsto x_{10} \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_{13} \longmapsto y_{13}$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1$$
$$y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(b) = \{c, x_{13}, x_8, x_4, x_{10}\} \subseteq S_\beta(b)$ and $S_\delta(v) \cap S_\beta(b) = \{c, y_{13}, y_8, y_4, y_6\} \subseteq S_\beta(b)$. Note further that both $P_{\alpha,\gamma}(c, x_{13}, x_8, x_4, x_{10})$ and $P_{\alpha,\gamma}(c, y_{13}, y_8, y_4, y_6)$ hold and that

$$\|\{c, x_{13}, x_8, x_4, x_{10}\} \cap \{c, y_{13}, y_8, y_4, y_6\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(b)\| \leq 14$ so that

$$\|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15. (1)$$

An application of the symetry Θ to (1) gives

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15. (2)$$

Now, we note that $x_5 \xrightarrow{\delta}$ —— y_9 since $x_5, y_9 \in S_\beta(c) \cap S_\alpha(e) \cap S_\gamma(d)$. Note further that $b, y_5, y_9, y_{10}, d, x_6, x_5, x_9 \in S_\beta(c)$. note also that the following hold:

$$\{y_5, y_9, y_{10}\} \cap \{x_6, x_5, x_9\} = \emptyset$$
$$P_{\alpha,\gamma}(b, y_5, y_9, y_{10}, d)$$
$$P_{\alpha,\gamma}(d, x_6, x_5, x_9, b)$$

Since $\|S_\beta(c)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (b, y_5, y_9, y_{10}, d, x_6, x_5, x_9)$$

to see that

$$\begin{aligned} &\text{either } N^2_{\delta,\gamma,\alpha}(b, y_5, y_9, y_{10}, d, x_6, x_5, x_9) \\ &\text{or } N^2_{\delta,\gamma,\alpha}(b, x_9, x_5, x_6, d, y_{10}, y_9, y_5). \end{aligned} \tag{3}$$

By an application of the symmetry Θ to (3), we see that

$$\begin{aligned} &\text{either } N^2_{\delta,\gamma,\beta}(c, y_9, y_1, y_4, b, x_4, x_7, x_5) \\ &\text{or } N^2_{\delta,\gamma,\beta}(c, x_5, x_7, x_4, b, y_4, y_1, y_9). \end{aligned} \tag{4}$$

By (3), we see that

$$\text{either } x_5 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_{10} \text{ and } x_6 \xrightarrow{\alpha} \!\!-\!\!-\!\!-\, y_9 \text{ or } x_5 \xrightarrow{\alpha} \!\!-\!\!-\!\!-\, y_{10} \text{ and } x_6 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_9. (5)$$

By (4), we see that

$$\text{either } x_7 \xrightarrow{\beta} \!\!-\!\!-\!\!-\, y_9 \text{ and } x_5 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_1 \text{ or } x_7 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_9 \text{ and } x_5 \xrightarrow{\beta} \!\!-\!\!-\!\!-\, y_1. (6)$$

Next, we note the following:

$$x_5 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_9 \text{ by (4).}$$

$$x_6 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_{10} \text{ by (3).}$$

$$x_7 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_1 \text{ by (4).}$$

$$\text{Either } x_3 \xrightarrow{\beta} \!\!-\!\!-\!\!-\, y_{10} \text{ or } x_3 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_{10} \text{ since } x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_3 \xrightarrow{\beta} \!\!-\!\!-\!\!-\, y_3 \text{ or } x_3 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_6 \xrightarrow{\beta} \!\!-\!\!-\!\!-\, y_3 \text{ or } x_6 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_3 \text{ since } x_6, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

Note further that $x_5, x_3, x_6, x_0, x_7, y_9, y_0, y_{10}, y_3, y_1 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_5, x_3, x_6, x_0, x_7\} \cap \{y_9, y_0, y_{10}, y_3, y_1\} = \emptyset$$

$$P_{\beta,\alpha}(x_5, x_3, x_6, x_0, x_7)$$

$$P_{\beta,\alpha}(y_9, y_0, y_{10}, y_3, y_1)$$

Since $\|S_\gamma(b)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 10 to see that

$$\begin{aligned} &\text{either } M^1_{\beta,\alpha,\delta}(x_6, x_0, x_7, x_5, x_3, y_{10}, y_3, y_1, y_9, y_0) \\ &\text{or } M^1_{\beta,\alpha,\delta}(x_0, x_6, x_3, x_5, x_7, y_9, y_1, y_3, y_{10}, y_0) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_0, x_6, x_3, x_5, x_7, y_{10}, y_3, y_1, y_9, y_0) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_3, x_6, x_0, x_7, x_5, y_{10}, y_0, y_9, y_1, y_3) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_3, x_5, x_7, x_0, x_6, y_1, y_9, y_0, y_{10}, y_3) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_0, x_7, x_5, x_3, x_6, y_9, y_1, y_3, y_{10}, y_0). \end{aligned} \tag{7}$$

By (7), we see that

$$\begin{aligned} &\text{if } x_0 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_0, \text{ then either } x_5 \xrightarrow{\beta} \!\!-\!\!-\!\!-\, y_0 \text{ and } x_0 \xrightarrow{\beta} \!\!-\!\!-\!\!-\, y_9 \\ &\text{or } x_5 \xrightarrow{\alpha} \!\!-\!\!-\!\!-\, y_0 \text{ and } x_0 \xrightarrow{\delta} \!\!-\!\!-\!\!-\, y_9. \end{aligned} \tag{8}$$

Again by (7), we see that

$$\text{if } x_0 \stackrel{\delta}{\to} \text{——}\, y_0, \text{ then either } x_7 \stackrel{\delta}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\alpha}{\to} \text{——}\, y_1$$
$$\text{or } x_7 \stackrel{\beta}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\beta}{\to} \text{——}\, y_1. \tag{9}$$

Again by (7), we see that

$$\text{if } x_0 \stackrel{\beta}{\to} \text{——}\, y_0, \text{ then either } x_5 \stackrel{\delta}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\delta}{\to} \text{——}\, y_9$$
$$\text{or } x_5 \stackrel{\alpha}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\alpha}{\to} \text{——}\, y_9.0 \tag{1}$$

Again by (7), we see that

$$\text{if } x_0 \stackrel{\alpha}{\to} \text{——}\, y_0, \text{ then } x_7 \stackrel{\alpha}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\beta}{\to} \text{——}\, y_1.(1)1$$

By an application of the symmetry Θ to (8), we see that

$$\text{if } x_0 \stackrel{\delta}{\to} \text{——}\, y_0, \text{ then either } x_7 \stackrel{\alpha}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\alpha}{\to} \text{——}\, y_1$$
$$\text{or } x_7 \stackrel{\beta}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\delta}{\to} \text{——}\, y_1.2 \tag{1}$$

From (9) and (12) and the fact that $x_0, y_0 \in S_\gamma(a)$, we conclude that

$$\text{either } x_0 \stackrel{\alpha}{\to} \text{——}\, y_0 \text{ or } x_0 \stackrel{\beta}{\to} \text{——}\, y_0.(1)3$$

By an application of the symmetry Θ to (10), we see that

$$\text{if } x_0 \stackrel{\alpha}{\to} \text{——}\, y_0, \text{ then either } x_7 \stackrel{\delta}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\delta}{\to} \text{——}\, y_1$$
$$\text{or } x_7 \stackrel{\beta}{\to} \text{——}\, y_0 \text{ and } x_0 \stackrel{\beta}{\to} \text{——}\, y_1. \tag{14}$$

From (11), (14), and (13), we conclude that

$$x_0 \stackrel{\beta}{\to} \text{——}\, y_0.(1)5$$

An application of the symmetry Θ to (15) gives

$$x_0 \stackrel{\alpha}{\to} \text{——}\, y_0.(1)6$$

Now, (15) and (16) together produce the desired contradiction. The proof is complete. □

Proposition 37. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^0_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_{11} = y_{11}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\alpha(c) = \{a, x_3, x_{10}, x_2, x_{12}\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_3, y_6, y_2, y_{12}\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(a, x_3, x_{10}, x_2, x_{12})$ and $P_{\beta,\gamma}(a, y_3, y_6, y_2, y_{12})$ hold and that

$$\|\{a, x_3, x_{10}, x_2, x_{12}\} \cap \{a, y_3, y_6, y_2, y_{12}\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$\|S_\gamma(c)\| \geq 15.(1)$$

Note also that $S_\delta(u) \cap S_\beta(a) = \{b, x_9, x_3, x_7, x_{12}\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_7, y_3, y_9, y_{12}\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_9, x_3, x_7, x_{12})$ and $P_{\alpha,\gamma}(b, y_7, y_3, y_9, y_{12})$ hold and that

$$\|\{b, x_9, x_3, x_7, x_{12}\} \cap \{b, y_7, y_3, y_9, y_{12}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$\|S_\gamma(a)\| \geq 15.(2)$$

Now, we note the following:

$$\begin{aligned}
x_4 &\overset{\delta}{\to} \text{——} y_4 \text{ since } x_4, y_4 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).\\
x_1 &\overset{\delta}{\to} \text{——} y_1 \text{ since } x_1, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).\\
x_8 &\overset{\delta}{\to} \text{——} y_8 \text{ since } x_8, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).\\
x_1 &\overset{\delta}{\to} \text{——} y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\alpha(b) \cap S_\beta(d).\\
\text{Either } x_7 &\overset{\alpha}{\to} \text{——} y_7 \text{ or } x_7 \overset{\delta}{\to} \text{——} y_7 \text{ since } x_7, y_7 \in S_\gamma(c) \cap S_\beta(d).\\
\text{Either } x_7 &\overset{\alpha}{\to} \text{——} y_1 \text{ or } x_7 \overset{\delta}{\to} \text{——} y_1 \text{ since } x_7, y_1 \in S_\gamma(c) \cap S_\beta(d).\\
\text{Either } x_4 &\overset{\alpha}{\to} \text{——} y_8 \text{ or } x_4 \overset{\delta}{\to} \text{——} y_8 \text{ since } x_4, y_8 \in S_\gamma(c) \cap S_\beta(b).\\
\text{Either } x_8 &\overset{\alpha}{\to} \text{——} y_4 \text{ or } x_8 \overset{\delta}{\to} \text{——} y_4 \text{ since } x_8, y_4 \in S_\gamma(c) \cap S_\beta(b).\\
\text{Either } x_1 &\overset{\beta}{\to} \text{——} y_8 \text{ or } x_1 \overset{\delta}{\to} \text{——} y_8 \text{ since } x_1, y_8 \in S_\gamma(c) \cap S_\alpha(a).\\
\text{Either } x_8 &\overset{\beta}{\to} \text{——} y_1 \text{ or } x_8 \overset{\delta}{\to} \text{——} y_1 \text{ since } x_8, y_1 \in S_\gamma(c) \cap S_\alpha(a).
\end{aligned}$$

Note further that $x_7, x_4, x_1, x_8, x_0, y_7, y_4, y_1, y_8, y_0 \in S_\gamma(c)$. Note also that the following hold:

$$\begin{gathered}
\{x_7, x_4, x_1, x_8, x_0\} \cap \{y_7, y_4, y_1, y_8, y_0\} = \emptyset\\
P_{\beta,\alpha}(x_7, x_4, x_1, x_8, x_0)\\
P_{\beta,\alpha}(y_7, y_4, y_1, y_8, y_0)
\end{gathered}$$

Since $\|S_\gamma(d)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_7, x_4, x_1, x_8, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_7, y_4, y_1, y_8, y_0)$$

to see that

$$\begin{aligned}
&\text{either } M^1_{\beta,\alpha,\delta}(x_4, x_7, x_0, x_8, x_1, y_4, y_1, y_8, y_0, y_7)\\
&\quad\text{or } M^1_{\beta,\alpha,\delta}(x_7, x_0, x_8, x_1, x_4, y_1, y_8, y_0, y_7, y_4).
\end{aligned} \tag{3}$$

Finally, we note the following:

$$x_4 \overset{\delta}{\to} \text{——} \, y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$
$$x_{10} \overset{\delta}{\to} \text{——} \, y_6 \text{ since } x_{10}, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$
$$x_6 \overset{\delta}{\to} \text{——} \, y_{10} \text{ since } x_6, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$
$$x_0 \overset{\beta}{\to} \text{——} \, y_0 \text{ by (3).}$$
$$\text{Either } x_2 \overset{\beta}{\to} \text{——} \, y_2 \text{ or } x_2 \overset{\delta}{\to} \text{——} \, y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b).$$

Note further that $x_4, x_2, x_{10}, x_0, x_6, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_4, x_2, x_{10}, x_0, x_6\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} = \emptyset$$
$$P_{\beta,\alpha}(x_4, x_2, x_{10}, x_0, x_6)$$
$$P_{\beta,\alpha}(y_4, y_2, y_6, y_0, y_{10})$$

Since $\|S_\gamma(a)\| \geq 15$ by (2), we may apply Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_4, x_2, x_{10}, x_0, x_6)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10})$$

to produce the desired contradiction. The proof is complete. $\square$

Proposition 38. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^0_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^0_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ where $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_{13} \longmapsto x_{13}$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_9 \longmapsto x_5 \longmapsto x_7 \longmapsto x_1$$
$$x_2 \longmapsto x_6 \longmapsto x_4 \longmapsto x_{10} \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_{13} \longmapsto y_{13}$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_9 \longmapsto y_5 \longmapsto y_7 \longmapsto y_1$$
$$y_2 \longmapsto y_6 \longmapsto y_4 \longmapsto y_{10} \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(b) = \{c, x_{10}, x_4, x_8, x_{13}\} \subseteq S_\beta(b)$ and $S_\delta(v) \cap S_\beta(b) = \{c, y_{10}, y_4, y_8, y_{13}\} \subseteq S_\beta(b)$. Note further that both $P_{\alpha,\gamma}(c, x_{10}, x_4, x_8, x_{13})$ and $P_{\alpha,\gamma}(c, y_{10}, y_4, y_8, y_{13})$ hold and that

$$\|\{c, x_{10}, x_4, x_8, x_{13}\} \cap \{c, y_{10}, y_4, y_8, y_{13}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(b)\| \leq 14$ so that

$$\|S_\gamma(b)\| \geq 15.(1)$$

Now, we note the following:

$$x_5 \overset{\delta}{\to} \text{——} \; y_5 \text{ since } x_5, y_5 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(c).$$
$$x_6 \overset{\delta}{\to} \text{——} \; y_6 \text{ since } x_6, y_6 \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c).$$
$$x_7 \overset{\delta}{\to} \text{——} \; y_7 \text{ since } x_7, y_7 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(d).$$
$$\text{Either } x_5 \overset{\beta}{\to} \text{——} \; y_7 \text{ or } x_5 \overset{\delta}{\to} \text{——} \; y_7 \text{ since } x_5, y_7 \in S_\gamma(b) \cap S_\alpha(e).$$
$$\text{Either } x_7 \overset{\beta}{\to} \text{——} \; y_5 \text{ or } x_7 \overset{\delta}{\to} \text{——} \; y_5 \text{ since } x_7, y_5 \in S_\gamma(b) \cap S_\alpha(e).$$
$$\text{Either } x_3 \overset{\beta}{\to} \text{——} \; y_3 \text{ or } x_3 \overset{\delta}{\to} \text{——} \; y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$
$$\text{Either } x_3 \overset{\beta}{\to} \text{——} \; y_6 \text{ or } x_3 \overset{\delta}{\to} \text{——} \; y_6 \text{ since } x_3, y_6 \in S_\gamma(b) \cap S_\alpha(d).$$
$$\text{Either } x_6 \overset{\beta}{\to} \text{——} \; y_3 \text{ or } x_6 \overset{\delta}{\to} \text{——} \; y_3 \text{ since } x_6, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$
$$\text{Either } x_5 \overset{\alpha}{\to} \text{——} \; y_6 \text{ or } x_5 \overset{\delta}{\to} \text{——} \; y_6 \text{ since } x_5, y_6 \in S_\gamma(b) \cap S_\beta(c).$$
$$\text{Either } x_6 \overset{\alpha}{\to} \text{——} \; y_5 \text{ or } x_6 \overset{\delta}{\to} \text{——} \; y_5 \text{ since } x_6, y_5 \in S_\gamma(b) \cap S_\beta(c).$$

Note further that $x_5, x_3, x_6, x_0, x_7, y_5, y_3, y_6, y_0, y_7 \in S_\gamma(b)$. Note also the following:

$$\{x_5, x_3, x_6, x_0, x_7\} \cap \{y_5, y_3, y_6, y_0, y_7\} = \emptyset$$
$$P_{\beta,\alpha}(x_5, x_3, x_6, x_0, x_7)$$
$$P_{\beta,\alpha}(y_5, y_3, y_6, y_0, y_7)$$

Since $\|S_\gamma(b)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_5, x_3, x_6, x_0, x_7)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_5, y_3, y_6, y_0, y_7)$$

to see that

$$\begin{aligned}
&\text{either } M^0_{\beta,\alpha,\delta}(x_0,x_7,x_5,x_3,x_6,y_0,y_7,y_5,y_3,y_6)\\
&\quad\text{or } M^0_{\beta,\alpha,\delta}(x_3,x_6,x_0,x_7,x_5,y_3,y_6,y_0,y_7,y_5)\\
&\quad\text{or } M^1_{\beta,\alpha,\delta}(x_6,x_0,x_7,x_5,x_3,y_6,y_0,y_7,y_5,y_3)\\
&\quad\text{or } M^1_{\beta,\alpha,\delta}(x_3,x_5,x_7,x_0,x_6,y_3,y_5,y_7,y_0,y_6)\\
&\quad\text{or } M^2_{\beta,\alpha,\delta}(x_3,x_6,x_0,x_7,x_5,y_6,y_3,y_5,y_7,y_0)\\
&\quad\text{or } M^2_{\beta,\alpha,\delta}(x_0,x_7,x_5,x_3,x_6,y_5,y_7,y_0,y_6,y_3). \qquad (2)
\end{aligned}$$

By (2), we see that

$$x_0 \overset{\delta}{\to} \text{——} y_5 \text{ and } x_5 \overset{\delta}{\to} \text{——} y_0.(3)$$

By an application of the symmetry Θ to (3), we see that

$$x_0 \overset{\delta}{\to} \text{——} y_7 \text{ and } x_7 \overset{\delta}{\to} \text{——} y_0.(4)$$

Now, from (4) and (2), we see that

$$M^0_{\beta,\alpha,\delta}(x_3,x_6,x_0,x_7,x_5,y_3,y_6,y_0,y_7,y_5).(5)$$

By (5), we see that

$$x_3 \overset{\alpha}{\to} \text{——} y_5, y_7.(6)$$

An application of the symmetry Θ to (6) gives

$$x_8 \overset{\beta}{\to} \text{——} y_7, y_1.(7)$$

An application of the symmetry Θ to (7) gives

$$x_3 \overset{\alpha}{\to} \text{——} y_1, y_9.(8)$$

By (6) and (8) we get $x_3 \overset{\alpha}{\to}$ —— y_7, y_1, which gives a monochromatic triangle, since $y_1 \overset{\alpha}{\to}$ —— y_7, thus producing the desired contradiction. The proof is complete. ☐

Proposition 39. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0,\ldots,x_{15}\}$ and $S_\delta(v) = \{y_0,\ldots,y_{15}\}$ with $B^0_{\alpha,\beta,\gamma}(x_0,\ldots,x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0,\ldots,y_{15})$ such that $S_\delta(u)\cap S_\delta(v) = \{a,b,c,d\}$ where $a = x_{14} = y_{13}$, $b = x_{15} = y_{14}$, $c = x_{11} = y_{15}$, and $d = x_{12} = y_5$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto b \longmapsto d \longmapsto c \longmapsto a$$
$$x_0 \longmapsto x_0$$
$$x_{13} \longmapsto x_{13}$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_9 \longmapsto x_5 \longmapsto x_7 \longmapsto x_1$$
$$x_2 \longmapsto x_6 \longmapsto x_4 \longmapsto x_{10} \longmapsto x_2$$
$$y_0 \longmapsto y_{10} \longmapsto y_1 \longmapsto y_4 \longmapsto y_0$$
$$y_2 \longmapsto y_3 \longmapsto y_{12} \longmapsto y_7 \longmapsto y_2$$
$$y_6 \longmapsto y_9 \longmapsto y_{11} \longmapsto y_8 \longmapsto y_6$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{13}, x_8, x_4, x_{10}\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_{13}, x_8, x_4, x_{10})$ and $P_{\alpha,\gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{13}, x_8, x_4, x_{10}\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$\|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.(1)$$

Repeated applications of the symmetry Θ to (1) give

$$\|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15, (2)$$

$$\|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15, (3)$$

and

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.(4)$$

First, we show that $x_5 \stackrel{\delta}{\to} \!\!\!-\!\!\!-\!\!\!-\; y_0$. Suppose not. Then $x_5 \stackrel{\beta}{\to} \!\!\!-\!\!\!-\!\!\!-\; y_0$ since $x_5, y_0 \in S_\alpha(d) \cap S_\gamma(a)$. Note also that $x_7 \stackrel{\delta}{\to} \!\!\!-\!\!\!-\!\!\!-\; y_9$ since $x_7, y_9 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(b)$. Note further that $a, y_9, y_0, y_6, b, x_5, x_7, x_4 \in S_\alpha(d)$. Note also that the following hold:

$$\{y_9, y_0, y_6\} \cap \{x_5, x_7, x_4\} = \emptyset$$
$$P_{\beta,\gamma}(a, y_9, y_0, y_6, b)$$
$$P_{\beta,\gamma}(b, x_5, x_7, x_4, a)$$

Since $\|S_\alpha(d)\| \geq 15$ by (3), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_9, y_0, y_6, b, x_5, x_7, x_4)$$

to get a contradiction. Thus, we have shown that

$$x_5 \overset{\delta}{\to} —— y_0, (5)$$

as desired.

By an application of the symmetry Θ to (5), we see that

$$x_7 \overset{\delta}{\to} —— y_{10}. (6)$$

Next, we note the following:

$$x_7 \overset{\delta}{\to} —— y_{10} \text{ by (6).}$$
$$x_6 \overset{\delta}{\to} —— y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$
$$x_5 \overset{\delta}{\to} —— y_{10} \text{ by (5).}$$
$$x_6 \overset{\delta}{\to} —— y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$
$$x_5 \overset{\delta}{\to} —— y_6 \text{ since } x_5, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$
$$\text{Either } x_7 \overset{\beta}{\to} —— y_6 \text{ or } x_7 \overset{\delta}{\to} —— y_6 \text{ since } x_7, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_7 \overset{\beta}{\to} —— y_0 \text{ or } x_7 \overset{\delta}{\to} —— y_0 \text{ since } x_7, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$
$$\text{Either } x_3 \overset{\beta}{\to} —— y_2 \text{ or } x_3 \overset{\delta}{\to} —— y_2 \text{ since } x_3, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$
$$\text{Either } x_3 \overset{\beta}{\to} —— y_6 \text{ or } x_3 \overset{\delta}{\to} —— y_6 \text{ since } x_3, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$
$$\text{Either } x_6 \overset{\alpha}{\to} —— y_4 \text{ or } x_6 \overset{\delta}{\to} —— y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b).$$
$$\text{Either } x_5 \overset{\alpha}{\to} —— y_4 \text{ or } x_5 \overset{\delta}{\to} —— y_4 \text{ since } x_5, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

Note further that$x_7, x_0, x_6, x_3, x_5, y_{10}, y_4, y_2, y_6, y_0 \in S_\gamma(a)$. Note also that the following hold:

$$\{x_7, x_0, x_6, x_3, x_5\} \cap \{y_{10}, y_4, y_2, y_6, y_0\} = \emptyset$$
$$P_{\beta,\alpha}(x_7, x_0, x_6, x_3, x_5)$$
$$P_{\beta,\alpha}(y_{10}, y_4, y_2, y_6, y_0)$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_7, x_0, x_6, x_3, x_5)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{10}, y_4, y_2, y_6, y_0)$$

to see that

$$\begin{aligned} &\text{either } M^2_{\beta,\alpha,\delta}(x_3, x_5, x_7, x_0, x_6, y_6, y_0, y_{10}, y_4, y_2) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_3, x_6, x_0, x_7, x_5, y_6, y_2, y_4, y_{10}, y_0). \end{aligned} \tag{7}$$

By (7), we have

$$x_0 \overset{\beta}{\to} —— y_2. (8)$$

Applying the symmetry Θ to (8), we get

$$x_0 \overset{\alpha}{\to} —— y_3. (9)$$

Applying the symmetry Θ to (9), we get

$$x_0 \overset{\beta}{\to} —— y_{12}. (1)0$$

But (8) and (10) give a monochromatic triangle, since $y_2 \overset{\beta}{\to}$ —— y_{12}, thus producing the desired contradiction. The proof is complete. ☐

Proposition 40. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{14} = y_{13}$, $b = x_{15} = y_{14}$, $c = x_{11} = y_{15}$, and $d = x_{12} = y_5$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto b \longmapsto d \longmapsto c \longmapsto a$$
$$x_0 \longmapsto x_0$$
$$x_{13} \longmapsto x_{13}$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_7 \longmapsto x_5 \longmapsto x_9 \longmapsto x_1$$
$$x_2 \longmapsto x_{10} \longmapsto x_4 \longmapsto x_6 \longmapsto x_2$$
$$y_0 \longmapsto y_{10} \longmapsto y_1 \longmapsto y_4 \longmapsto y_0$$
$$y_2 \longmapsto y_3 \longmapsto y_{12} \longmapsto y_7 \longmapsto y_2$$
$$y_6 \longmapsto y_9 \longmapsto y_{11} \longmapsto y_8 \longmapsto y_6$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{13}, x_8, x_4, x_6\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_{13}, x_8, x_4, x_6)$ and $P_{\alpha,\gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{13}, x_8, x_4, x_6\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$\|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15. (1)$$

Repeated applications of the symmetry Θ to (1) give

$$\|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15, (2)$$

$$\|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15, (3)$$

and

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15. (4)$$

First, we show that $x_9 \overset{\delta}{\to}$ —— y_0. Suppose not. Then $x_9 \overset{\beta}{\to}$ —— y_0 since $x_9, y_0 \in S_\alpha(d) \cap S_\gamma(c)$. Note also that $x_1 \overset{\delta}{\to}$ —— y_9 since $x_1, y_9 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(b)$. Note further that

$a, y_9, y_0, y_6, b, x_9, x_1, x_4 \in S_\alpha(d)$. Note also that the following hold:

$$\{y_9, y_0, y_6\} \cap \{x_9, x_1, x_4\} = \emptyset$$
$$P_{\beta,\gamma}(a, y_9, y_0, y_6, b)$$
$$P_{\beta,\gamma}(b, x_9, x_1, x_4, a)$$

Since $\|S_\alpha(d)\| \geq 15$ by (3), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_9, y_0, y_6, b, x_9, x_1, x_4)$$

to get a contradiction. Thus, we have shown that

$$x_9 \xrightarrow{\delta} \text{——} y_0, (5)$$

as desired.

By an application of the symmetry Θ to (5), we see that

$$x_1 \xrightarrow{\delta} \text{——} y_{10}.(6)$$

Next, we note the following:

$$x_9 \xrightarrow{\delta} \text{——} y_0 \text{ by (5).}$$
$$x_{10} \xrightarrow{\delta} \text{——} y_2 \text{ since } S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$
$$x_1 \xrightarrow{\delta} \text{——} y_{10} \text{ by (6).} x_9 \xrightarrow{\delta} \text{——} y_6 \text{ since } S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$
$$x_{10} \xrightarrow{\delta} \text{——} y_6 \text{ since } S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$
$$x_3 \xrightarrow{\beta} \text{——} y_6 \text{ or } x_3 \xrightarrow{\delta} \text{——} y_6 \text{ since } x_3, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$
$$x_3 \xrightarrow{\beta} \text{——} y_2 \text{ or } x_3 \xrightarrow{\delta} \text{——} y_2 \text{ since } x_3, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$
$$x_1 \xrightarrow{\beta} \text{——} y_0 \text{ or } x_1 \xrightarrow{\delta} \text{——} y_0 \text{ since } x_1, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$
$$x_1 \xrightarrow{\beta} \text{——} y_6 \text{ or } x_1 \xrightarrow{\delta} \text{——} y_6 \text{ since } x_1, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$
$$x_9 \xrightarrow{\alpha} \text{——} y_4 \text{ or } x_9 \xrightarrow{\delta} \text{——} y_4 \text{ since } x_9, y_4 \in S_\gamma(a) \cap S_\beta(b).$$
$$x_{10} \xrightarrow{\alpha} \text{——} y_4 \text{ or } x_{10} \xrightarrow{\delta} \text{——} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

Note further that $x_9, x_0, x_{10}, x_3, x_1, y_0, y_6, y_2, y_4, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_9, x_0, x_{10}, x_3, x_1\} \cap \{y_0, y_6, y_2, y_4, y_{10}\} = \emptyset$$
$$P_{\beta,\alpha}(x_9, x_0, x_{10}, x_3, x_1)$$
$$P_{\beta,\alpha}(y_0, y_6, y_2, y_4, y_{10})$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_{10}, x_3, x_1)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_6, y_2, y_4, y_{10})$$

to see that

$$\begin{aligned} &\text{either } M^2_{\beta,\alpha,\delta}(x_0, x_{10}, x_3, x_1, x_9, y_6, y_2, y_4, y_{10}, y_0) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_0, x_9, x_1, x_3, x_{10}, y_6, y_0, y_{10}, y_4, y_2). \end{aligned} \tag{7}$$

By (7), we see that

$$x_3 \overset{\beta}{\to} —— y_2.(8)$$

Applying the symmetry Θ to (8) gives

$$x_8 \overset{\alpha}{\to} —— y_3.(9)$$

Applying the symmetry Θ to (9) gives

$$x_3 \overset{\beta}{\to} —— y_{12}.(1)0$$

But (8) and (10) give a monochromatic triangle, since $y_2 \overset{\beta}{\to} —— y_{12}$, thus producing the desired contradiction. The proof is complete. □

Proposition 41. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_5 = y_5$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto b \longmapsto d \longmapsto c \longmapsto a$$
$$x_0 \longmapsto x_{10} \longmapsto x_1 \longmapsto x_4 \longmapsto x_0$$
$$x_2 \longmapsto x_3 \longmapsto x_{12} \longmapsto x_7 \longmapsto x_2$$
$$x_6 \longmapsto x_9 \longmapsto x_{11} \longmapsto x_8 \longmapsto x_6$$
$$y_0 \longmapsto y_{10} \longmapsto y_1 \longmapsto y_4 \longmapsto y_0$$
$$y_2 \longmapsto y_3 \longmapsto y_{12} \longmapsto y_7 \longmapsto y_2$$
$$y_6 \longmapsto y_9 \longmapsto y_{11} \longmapsto y_8 \longmapsto y_6$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{12}, x_9, x_3, x_7\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_{12}, x_9, x_3, x_7)$ and $P_{\alpha,\gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{12}, x_9, x_3, x_7\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$\|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.(1)$$

Repeated applications of the symmetry Θ to (1) give

$$\|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15, (2)$$

$$\|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15, (3)$$

and

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.(4)$$

Now, we show that $x_4 \xrightarrow{\delta} y_4$. Suppose not. Then $x_4 \xrightarrow{\alpha} y_4$ since $x_4, y_4 \in S_\gamma(a) \cap S_\beta(b)$. We also note the following:

$$x_6 \xrightarrow{\delta} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\alpha(x_4) \cap S_\beta(b).$$

$$x_2 \xrightarrow{\delta} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(d).$$

$$x_4 \xrightarrow{\delta} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(y_4) \cap S_\beta(b).$$

$$x_6 \xrightarrow{\delta} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$\text{Either } x_0 \xrightarrow{\beta} y_6 \text{ or } x_0 \xrightarrow{\delta} y_6 \text{ since } x_0, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_0 \xrightarrow{\beta} y_0 \text{ or } x_0 \xrightarrow{\delta} y_0 \text{ since } x_0, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_6 \xrightarrow{\beta} y_0 \text{ or } x_6 \xrightarrow{\delta} y_0 \text{ since } x_6, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_{10} \xrightarrow{\alpha} y_{10} \text{ or } x_{10} \xrightarrow{\delta} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\beta(c).$$

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_{10}, y_4, y_2, y_6, y_0 \in S_\gamma(a)$. Note also that the following hold:

$$\{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_{10}, y_4, y_2, y_6, y_0\} = \emptyset$$

$$P_{\beta,\alpha}(x_0, x_6, x_2, x_4, x_{10})$$

$$P_{\beta,\alpha}(y_{10}, y_4, y_2, y_6, y_0)$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{10}, y_4, y_2, y_6, y_0)$$

to get a contradiction. Thus, we have

$$x_4 \xrightarrow{\delta} y_4, (5)$$

as desired.

By repeated applications of the symmetry Θ to (5), we see that

$$x_0 \xrightarrow{\delta} y_0, (6)$$

$$x_{10} \xrightarrow{\delta} y_{10}, (7)$$

and

$$x_1 \xrightarrow{\delta} y_1.(8)$$

Now, note that $x_0 \xrightarrow{\delta} y_0$ by (6). Note further that $a, y_9, y_0, y_6, b, x_6, x_0, x_9 \in S_\alpha(d)$ Note also the following:

$$\{y_9, y_0, y_6\} \cap \{x_6, x_0, x_9\} = \emptyset$$

$$P_{\beta,\gamma}(a, y_9, y_0, y_6, b)$$

$$P_{\beta,\gamma}(b, x_6, x_0, x_9, a)$$

Since $\|S_\alpha(d)\| \geq 15$ by (3), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_9, y_0, y_6, b, x_6, x_0, x_9)$$

to see that

$$\text{either } N^2_{\delta,\gamma,\beta}(a, y_9, y_0, y_6, b, x_6, x_0, x_9)$$

(9)
$$\text{or } N^2_{\delta,\gamma,\beta}(a, x_9, x_0, x_6, b, y_6, y_0, y_9).$$

By (9), we have

$$\text{either } x_6 \overset{\beta}{\to} \text{---} y_0 \text{ or } x_0 \overset{\beta}{\to} \text{---} y_6.(1)0$$

Finally, we note the following:

$$x_0 \overset{\delta}{\to} \text{---} y_0 \text{ by (6).}$$

$$x_6 \overset{\delta}{\to} \text{---} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$x_4 \overset{\delta}{\to} \text{---} y_4 \text{ by (5).}$$

$$x_{10} \overset{\delta}{\to} \text{---} y_{10} \text{ by (7).}$$

$$\text{Either } x_0 \overset{\beta}{\to} \text{---} y_6 \text{ or } x_0 \overset{\delta}{\to} \text{---} y_6 \text{ since } x_0, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_6 \overset{\beta}{\to} \text{---} y_0 \text{ or } x_6 \overset{\delta}{\to} \text{---} y_0 \text{ since } x_6, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_6 \overset{\beta}{\to} \text{---} y_2 \text{ or } x_6 \overset{\delta}{\to} \text{---} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$

$$\text{Either } x_2 \overset{\beta}{\to} \text{---} y_6 \text{ or } x_2 \overset{\delta}{\to} \text{---} y_6 \text{ since } x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$

$$\text{Either } x_6 \overset{\alpha}{\to} \text{---} y_4 \text{ or } x_6 \overset{\delta}{\to} \text{---} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

$$\text{Either } x_4 \overset{\alpha}{\to} \text{---} y_6 \text{ or } x_4 \overset{\delta}{\to} \text{---} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\beta(b).$$

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_0, y_6, y_2, y_4, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_0, y_6, y_2, y_4, y_{10}\} = \emptyset$$

$$P_{\beta,\alpha}(x_0, x_6, x_2, x_4, x_{10})$$

$$P_{\beta,\alpha}(y_0, y_6, y_2, y_4, y_{10})$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_6, y_2, y_4, y_{10})$$

to produce, with the help of (10), a contradiction. The proof is complete. □

Proposition 42. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ, and δ. Suppose that $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_{11} = y_5$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{12}, x_9, x_3, x_7\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_{12}, x_9, x_3, x_7)$ and $P_{\alpha,\gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{12}, x_9, x_3, x_7\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$\|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.(1)$$

Note also that $S_\delta(u) \cap S_\alpha(b) = \{d, x_7, x_5, x_2, x_{12}\} \subseteq S_\alpha(b)$ and $S_\delta(v) \cap S_\alpha(b) = \{d, y_2, y_{12}, y_{11}, y_7\} \subseteq S_\alpha(b)$. Note further that both $P_{\beta,\gamma}(d, x_7, x_5, x_2, x_{12})$ and $P_{\beta,\gamma}(d, y_2, y_{12}, y_{11}, y_7)$ hold and that

$$\|\{d, x_7, x_5, x_2, x_{12}\} \cap \{d, y_2, y_{12}, y_{11}, y_7\}\| = \|\{d\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(b)\| \leq 14$ so that

$$\|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15.(2)$$

Note also that $S_\delta(u) \cap S_\alpha(c) = \{a, x_{12}, x_2, x_6, x_3\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_{12}, y_2, y_6, y_3\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(a, x_{12}, x_2, x_6, x_3)$ and $P_{\beta,\gamma}(a, y_{12}, y_2, y_6, y_3)$ hold and that

$$\|\{a, x_{12}, x_2, x_6, x_3\} \cap \{a, y_{12}, y_2, y_6, y_3\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$\|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.(3)$$

Note also that $S_\delta(u) \cap S_\beta(d) = \{c, x_{12}, x_1, x_7, x_6\} \subseteq S_\beta(d)$ and $S_\delta(v) \cap S_\beta(d) = \{c, y_2, y_8, y_7, y_3\} \subseteq S_\beta(d)$. Note further that both $P_{\alpha,\gamma}(c, x_{12}, x_1, x_7, x_6)$ and $P_{\alpha,\gamma}(c, y_2, y_8, y_7, y_3)$ hold and that

$$\|\{c, x_{12}, x_1, x_7, x_6\} \cap \{c, y_2, y_8, y_7, y_3\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(d)\| \leq 14$ so that

$$\|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15.(4)$$

Note that $x_6, x_2, x_4, x_{10}, x_0, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_6, x_2, x_4, x_{10}, x_0\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} = \emptyset$$
$$P_{\beta,\alpha}(x_6, x_2, x_4, x_{10}, x_0)$$
$$P_{\beta,\alpha}(y_4, y_2, y_6, y_0, y_{10})$$

Noting also that $\|S_\gamma(a)\| \geq 15$ by (1), we will apply Lemma 14 several times in what follows with

$$(u_3, u_4, u_0, u_1, u_2) = (x_6, x_2, x_4, x_{10}, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10}).$$

We now show that $x_2 \overset{\delta}{\to} \text{——}\, y_6$. Suppose not. Then $x_2 \overset{\beta}{\to} \text{——}\, y_6$ since $x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c)$. We also note the following:

$$x_6 \overset{\delta}{\to} \text{——}\, y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$
$$x_4 \overset{\delta}{\to} \text{——}\, y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

Also, $x_{10} \overset{\beta}{\to} \text{——}\, y_6$ since otherwise, by Lemma 14, in view of the fact that $x_{10}, y_6 \in S_\gamma(a) \cap S_\alpha(d)$, we would have $y_6 \overset{\delta}{\to} \text{——}\, x_6, x_4, x_{10}$, so that $y_6 \overset{\delta}{\to} \text{——}\, x_2, x_0$, also, which would contradict the fact that $x_2 \overset{\beta}{\to} \text{——} y_6$. Now, note the following:

$$x_2 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(y_6).$$

$$x_{10} \overset{\delta}{\to} \text{——} y_0 \text{ since } x_{10}, y_0 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(y_6).$$

$$x_6 \overset{\delta}{\to} \text{——} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$

$$\text{Either } x_4 \overset{\beta}{\to} \text{——} y_0 \text{ or } x_4 \overset{\delta}{\to} \text{——} y_0 \text{ since } x_4, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

Note further that $x_6, x_2, x_4, x_{10}, x_0, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_6, x_2, x_4, x_{10}, x_0\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} = \emptyset$$
$$P_{\beta,\alpha}(x_6, x_2, x_4, x_{10}, x_0)$$
$$P_{\beta,\alpha}(y_4, y_2, y_6, y_0, y_{10})$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_6, x_2, x_4, x_{10}, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10})$$

to get a contradiction. Thus, we have

$$x_2 \overset{\delta}{\to} \text{——} y_6, (5)$$

as desired.

Now, we note the following:

$$x_2 \overset{\delta}{\to} \text{——} y_6 \text{ by (5).}$$

$$x_6 \overset{\delta}{\to} \text{——} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$x_4 \overset{\delta}{\to} \text{——} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

Thus, by Lemma 14, we have

$$x_6, x_2, x_4, x_{10}, x_0 \overset{\delta}{\to} \text{——} y_6.(6)$$

Next, we note the following:

$$x_{10} \overset{\delta}{\to} \text{——} y_6 \text{ by (6).}$$

$$x_{10} \overset{\delta}{\to} \text{——} y_9 \text{ since } x_{10}, y_9 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(b).$$

Note further that $a, y_9, y_0, y_6, b, x_4, x_{10}, x_3 \in S_\alpha(d)$. Note also that the following hold:

$$\{y_9, y_0, y_6\} \cap \{x_4, x_{10}, x_3\} = \emptyset$$
$$P_{\beta,\gamma}(a, y_9, y_0, y_6, b)$$
$$P_{\beta,\gamma}(b, x_4, x_{10}, x_3, a)$$

Since $\|S_\alpha(d)\| \geq 15$ by (4), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (a, y_9, y_0, y_6, b, x_4, x_{10}, x_3)$$

to see that

$$\begin{aligned} &\text{either } N^0_{\delta,\gamma,\beta}(a, y_9, y_0, y_6, b, x_4, x_{10}, x_3) \\ &\quad\text{or } N^1_{\delta,\gamma,\beta}(a, y_9, y_0, y_6, b, x_4, x_{10}, x_3) \\ &\quad\text{or } N^1_{\delta,\gamma,\beta}(a, x)3, x_{10}, x_4, b, y_6, y_0, y_9). \end{aligned} \tag{7}$$

Next, we note the following:

$$x_6 \overset{\delta}{\rightarrow} \text{——}\, y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$
$$x_4 \overset{\delta}{\rightarrow} \text{——}\, y_0 \text{ by (7).}$$
$$x_0, x_6, x_2, x_4, x_{10} \overset{\delta}{\rightarrow} \text{——}\, y_6 \text{ by (6).}$$
$$x_{10} \overset{\beta}{\rightarrow} \text{——}\, y_0 \text{ by (7).}$$
$$\text{Either } x_6 \overset{\alpha}{\rightarrow} \text{——}\, y_4 \text{ or } x_6 \overset{\delta}{\rightarrow} \text{——}\, y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} = \emptyset$$
$$P_{\beta,\alpha}(x_0, x_6, x_2, x_4, x_{10})$$
$$P_{\beta,\alpha}(y_4, y_2, y_6, y_0, y_{10})$$

Since $\|S_\gamma(a)\| \geq 15$ by (1), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10})$$

to see that

$$\begin{aligned} &\text{either } M^2_{\beta,\alpha,\delta}(x_2, x_4, x_{10}, x_0, x_6, y_6, y_0, y_{10}, y_4, y_2) \\ &\text{or } M^2_{\beta,\alpha,\delta}(x_2, x_6, x_0, x_{10}, x_4, y_6, y_2, y_4, y_{10}, y_0). \end{aligned} \tag{8}$$

Next, we note the following:

$$x_4 \overset{\delta}{\rightarrow} \text{——}\, y_0 \text{ by (8).}$$
$$x_1 \overset{\delta}{\rightarrow} \text{——}\, y_8 \text{ since } x_1, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$
$$x_0 \overset{\delta}{\rightarrow} \text{——}\, y_4 \text{ by (8).}$$
$$x_7 \overset{\delta}{\rightarrow} \text{——}\, y_7 \text{ since } x_7, y_7 \in S_\gamma(c) \cap S_\alpha(b) \cap S_\beta(d).$$
$$x_8 \overset{\delta}{\rightarrow} \text{——}\, y_8 \text{ since } x_8, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$
$$x_0 \overset{\alpha}{\rightarrow} \text{——}\, y_0 \text{ by (8).}$$
$$\text{Either } x_1 \overset{\beta}{\rightarrow} \text{——}\, y_1 \text{ or } x_1 \overset{\delta}{\rightarrow} \text{——}\, y_1 \text{ since } x_1, y_1 \in S_\gamma(c) \cap S_\alpha(a).$$
$$\text{Either } x_4 \overset{\alpha}{\rightarrow} \text{——}\, y_8 \text{ or } x_4 \overset{\delta}{\rightarrow} \text{——}\, y_8 \text{ since } x_4, y_8 \in S_\gamma(c) \cap S_\beta(b).$$
$$\text{Either } x_4 \overset{\alpha}{\rightarrow} \text{——}\, y_4 \text{ or } x_4 \overset{\delta}{\rightarrow} \text{——}\, y_4 \text{ since } x_4, y_4 \in S_\gamma(c) \cap S_\beta(b).$$
$$\text{Either } x_1 \overset{\alpha}{\rightarrow} \text{——}\, y_7 \text{ or } x_1 \overset{\delta}{\rightarrow} \text{——}\, y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\beta(d).$$
$$\text{Either } x_8 \overset{\alpha}{\rightarrow} \text{——}\, y_4 \text{ or } x_8 \overset{\delta}{\rightarrow} \text{——}\, y_4 \text{ since } x_8, y_4 \in S_\gamma(c) \cap S_\beta(b).$$
$$\text{Either } x_7 \overset{\alpha}{\rightarrow} \text{——}\, y_8 \text{ or } x_7 \overset{\delta}{\rightarrow} \text{——}\, y_8 \text{ since } x_7, y_8 \in S_\gamma(c) \cap S_\beta(d).$$

Note further that $x_4, x_1, x_8, x_0, x_7, y_0, y_8, y_1, y_4, y_7 \in S_\gamma(c)$. Note also that the following hold:

$$\begin{gathered}\{x_4, x_1, x_8, x_0, x_7\} \cap \{y_0, y_8, y_1, y_4, y_7\} = \emptyset \\ P_{\beta,\alpha}(x_4, x_1, x_8, x_0, x_7) \\ P_{\beta,\alpha}(y_0, y_8, y_1, y_4, y_7)\end{gathered}$$

Since $\|S_\gamma(c)\| \geq 15$ by (3), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_4, x_1, x_8, x_0, x_7)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_8, y_1, y_4, y_7)$$

to see that

$$\begin{gathered}\text{either } M^2_{\beta,\alpha,\delta}(x_0, x_7, x_4, x_1, x_8, y_4, y_7, y_0, y_8, y_1) \\ \text{or } M^2_{\beta,\alpha,\delta}(x_1, x_4, x_7, x_0, x_8, y_8, y_0, y_7, y_4, y_1).\end{gathered} \tag{9}$$

Now, note that $x_8 \overset{\delta}{\to} \text{——}\ y_8, y_1$ by (9). Note further that $c, y_{11}, y_1, y_8, d, x_1, x_8, x_5 \in S_\alpha(a)$. Note also that the following hold:

$$\begin{gathered}\{y_{11}, y_1, y_8\} \cap \{x_1, x_8, x_5\} = \emptyset \\ P_{\beta,\gamma}(c, y_{11}, y_1, y_8, d) \\ P_{\beta,\gamma}(d, x_1, x_8, x_5, c)\end{gathered}$$

Since $\|S_\alpha(a)\| \geq 15$ by (1), we may apply Lemma 12 with

$$(u_1, \ldots, u_8) = (c, y_{11}, y_1, y_8, d, x_1, x_8, x_5)$$

to see that

$$N^2_{\delta,\gamma,\beta}(c, y_{11}, y_1, y_8, d, x_1, x_8, x_5).(1)0$$

From (10), we get

$$x_1 \overset{\beta}{\to} \text{——}\ y_1.(11)$$

From (11) and (9), we see that

$$M^2_{\beta,\alpha,\delta}(x_0, x_7, x_4, x_1, x_8, y_4, y_7, y_0, y_8, y_1).(1)2$$

From (12), we get

$$x_4 \overset{\delta}{\to} \text{——}\ y_4.(1)3$$

From (13) and (8), we see that

$$M^2_{\beta,\alpha,\delta}(x_2, x_6, x_0, x_{10}, x_4, y_6, y_2, y_4, y_{10}, y_0).(1)4$$

Finally, we note the following:

$$x_0 \xrightarrow{\delta} y_1 \text{ by (12).}$$
$$x_{10} \xrightarrow{\delta} y_9 \text{ by (7).}$$
$$x_3 \xrightarrow{\delta} y_0 \text{ by (7).}$$
$$x_3 \xrightarrow{\delta} y_9 \text{ by (7).}$$
$$x_0 \xrightarrow{\alpha} y_0 \text{ by (12).}$$
$$x_0 \xrightarrow{\alpha} y_{10} \text{ by (14).}$$
$$x_{10} \xrightarrow{\beta} y_0 \text{ by (14).}$$
$$x_1 \xrightarrow{\beta} y_1 \text{ by (11).}$$
$$x_1 \xrightarrow{\beta} y_0 \text{ by (12).}$$

Note further $x_9, x_0, x_{10}, x_3, x_1, y_3, y_1, y_9, y_0, y_{10} \in S_\gamma(b)$. Note also that the following hold:

$$\{x_9, x_0, x_{10}, x_3, x_1\} \cap \{y_3, y_1, y_9, y_0, y_{10}\} = \emptyset$$
$$P_{\beta,\alpha}(x_9, x_0, x_{10}, x_3, x_1)$$
$$P_{\beta,\alpha}(y_3, y_1, y_9, y_0, y_{10})$$

Since $\|S_\gamma(b)\| \geq 15$ by (2), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_{10}, x_3, x_1)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_3, y_1, y_9, y_0, y_{10})$$

to produce the desired contradiction. The proof is complete. $\square$

Theorem 12. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 4$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 4$. There are, up to isomorphism, exactly three possibilities consistent with Proposition 28. Let $S_\delta(u) \cap S_\delta(v) = \{w_0, w_1, w_2, w_3\}$.

The first possibility is $w_0 \xrightarrow{\alpha} w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\alpha} w_3 \xrightarrow{\alpha} w_0$ and $w_0 \xrightarrow{\beta} w_2$ and $w_1 \xrightarrow{\beta} w_3$. In this case, we get a contradiction by Proposition 18.

The second possibility is $w_0 \xrightarrow{\alpha} w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\alpha} w_3 \xrightarrow{\alpha} w_0$ and $w_0 \xrightarrow{\beta} w_2$ and $w_1 \xrightarrow{\gamma} w_3$. In this case, we get a contradiction by Proposition 29 and Proposition 30.

Finally, we consider the third possibility, namely, $w_0 \xrightarrow{\alpha} w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\alpha} w_3$ and $w_1 \xrightarrow{\beta} w_3 \xrightarrow{\beta} w_0 \xrightarrow{\beta} w_2$. In this case, we get a contradiction by Proposition 31, Proposition 32, Proposition 33, Proposition 34, Proposition 35, Proposition 36, Proposition 37, Proposition 38, Proposition 39, Proposition 40, Proposition 41, and Proposition 42.

The proof is complete. $\square$

4.13. 3.13. Attaching Sets of Cardinality 3.

Proposition 43. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 3$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$(56)\quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [2,0,0].$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\big\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\big\| \leq R(3,3;2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η, so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$(57)\quad S_\delta(u) \cap S_\delta(v) = \{x\} \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big),$$

so that

$$(58)\quad 3 = \big\|S_\delta(u) \cap S_\delta(v)\big\| = 1 + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\big\|.$$

Thus, we have

$$(59)\quad \big(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|\big) \in [2,0,0] \cup [1,1,0].$$

The proposition now follows by an application of Proposition 4. □

Theorem 13. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 3$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 3$. But there are no possibilities consistent with Proposition 43. □

5. The Global Arguments

Theorem 14. *Let V be the vertex set of a complete graph with a good edge coloring with four colors. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $u \neq v$ be such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ for some color δ. Then $\|S_\delta(u) \cap S_\delta(v)\| \in \{0,1,2,5\}$. In addition, if $\|S_\delta(u) \cap S_\delta(v)\| = 5$, then there exist $x_0, \ldots, x_{15} \in S_\delta(u)$ and $y_0, \ldots, y_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{11,12,13,14,15\}$ and some $j \in \{0,1\}$ and colors α, β, and γ, such that $B^j_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^j_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ with*

$$S_\delta(u) \cap S_\delta(v) = \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\} = \{y_{11}, y_{12}, y_{13}, y_{14}, y_{15}\}.$$

Furthermore, if $\|S_\delta(u) \cap S_\delta(v)\| = 5$, then for each $w \in S_\delta(u) \cap S_\delta(v)$, we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted.

Proof. This is a trivial consequence of Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5, Theorem 6, Theorem 7, Theorem 8, Theorem 9, Theorem 10, Theorem 11, and Theorem 12. □

Theorem 15. *Let V be the vertex set of a complete graph with a good edge coloring, colored with four colors. Suppose that $\|V\| = 62$ and let $u, v, w, x, y \in V$ be pairwise distince vertices which satisfy $u, v, w \xrightarrow{\delta}$ —— x, y for some color δ. Then, for some $z \in \{u, v, w, x, y\}$ we have $\|S_\delta(z)\| \leq 15$.*

Proof. Suppose not. Then $\|S_\delta(z)\| = 16$ for all $z \in \{u, v, w, x, y\}$. Thus, by Theorem 13, we have $\|S_\delta(x) \cap S_\delta(y)\| \in \{0, 1, 2, 5\}$, so that, since $u, v, w \in S_\delta(x) \cap S_\delta(y)$, we must have

$$\|S_\delta(x) \cap S_\delta(y)\| = 5.$$

Next, we show that $\|S_\delta(u) \cap S_\delta(v)\| = \|S_\delta(u) \cap S_\delta(w)\| = \|S_\delta(v) \cap S_\delta(w)\| = 5$. Suppose not. Then we may, without loss of generality, suppose that $\|S_\delta(u) \cap S_\delta(v)\| \neq 5$, so that, by Theorem 13, we have $\|S_\delta(u) \cap S_\delta(v)\| \leq 2$. But, $x, y \in S_\delta(u) \cap S_\delta(v)$, so that we must have $S_\delta(u) \cap S_\delta(v) = \{x, y\} \subseteq S_\delta(w)$. But then we have $\big(S_\delta(u) \sim S_\delta(w)\big)\big(S_\delta(v) \sim S_\delta(w)\big) = \emptyset$. Thus, we see that

$$\begin{aligned}
62 &= \big\|V\big\| \\
&\geq \big\|S_\delta(u) \cup S_\delta(v) \cup S_\delta(w) \cup S_\delta(x) \cup S_\delta(y)\big\| \\
&= \big\|\big(S_\delta(x) \sim S_\delta(y)\big) \uplus \big(S_\delta(y) \sim S_\delta(x)\big) \uplus \big(S_\delta(x) \cap S_\delta(y)\big) \uplus \\
&\qquad \big(S_\delta(u) \sim S_\delta(w)\big) \uplus \big(S_\delta(v) \sim S_\delta(w)\big) \uplus S_\delta(w)\big\| \\
&= \big\|S_\delta(x) \sim S_\delta(y)\big\| + \big\|S_\delta(y) \sim S_\delta(x)\big\| + \big\|S_\delta(x) \cap S_\delta(y)\big\| + \\
&\qquad \big\|S_\delta(u) \sim S_\delta(w)\big\| + \big\|S_\delta(v) \sim S_\delta(w)\big\| + \big\|S_\delta(w)\big\| \\
&= 11 + 11 + 5 + \big\|S_\delta(u) \sim S_\delta(w)\big\| + \big\|S_\delta(v) \sim S_\delta(w)\big\| + 16 \\
&\geq 11 + 11 + 5 + 11 + 11 + 16 \\
&= 65,
\end{aligned}$$

which is a contradiction. Thus, we have

$$\|S_\delta(u) \cap S_\delta(v)\| = \|S_\delta(u) \cap S_\delta(w)\| = \|S_\delta(v) \cap S_\delta(w)\| = 5,$$

as desired.

We may now apply Theorem 13 to each pair in $\{u, v, w\}$ to see that there exist colors $\gamma, \gamma', \gamma'' \neq \delta$ such that

$$S_\delta(u) \cap S_\delta(v) \text{ contains no edges of color } \gamma \text{ and } \|S_\gamma(x)\| \leq 14,$$
$$S_\delta(u) \cap S_\delta(w) \text{ contains no edges of color } \gamma' \text{ and } \|S_{\gamma'}(x)\| \leq 14,$$

and

$$S_\delta(v) \cap S_\delta(w) \text{ contains no edges of color } \gamma'' \text{ and } \|S_{\gamma''}(x)\| \leq 14.$$

By Proposition 1, we see that $\gamma = \gamma' = \gamma''$.

Thus, there exists some $\gamma \neq \delta$, such that $\|S_\gamma(x)\| \leq 14$ and

$$\begin{aligned}
\emptyset &= (S_\delta(u) \cap S_\gamma(x)) \cap (S_\delta(v) \cap S_\gamma(x)) \\
&= (S_\delta(u) \cap S_\gamma(x)) \cap (S_\delta(v) \cap S_\gamma(x)) \\
&= (S_\delta(u) \cap S_\gamma(x)) \cap (S_\delta(v) \cap S_\gamma(x)).
\end{aligned}$$

Next, we show that $\|S_\delta(z) \cap S_\gamma(x)\| \leq 4$ for some $z \in \{u, v, w\}$. Suppose not. Then $\|S_\delta(z) \cap S_\gamma(x)\| = 5$ for all $z \in \{u, v, w\}$. Thus, since

$$\biguplus_{z \in \{u, v, w\}} \big(S_\delta(z) \cap S_\gamma(x)\big) \subseteq S_\gamma(x),$$

we must have $5 + 5 + 5 \leq 14$, which is impossible.

Thus, we may suppose, without loss of generality, that $\|S_\delta(u) \cap S_\gamma(x)\| \leq 4$. Let α and β be the two other colors. Then, since $u \overset{\delta}{\longrightarrow} x$, we must have

$$\begin{aligned}
16 &= \big\|S_\delta(u)\big\| \\
&= \big\|\{x\} \uplus \big(S_\delta(u) \cap S_\alpha(x)\big) \uplus \big(S_\delta(u) \cap S_\beta(x)\big) \uplus \big(S_\delta(u) \cap S_\gamma(x)\big)\big\| \\
&= \big\|\{x\}\big\| + \big\|S_\delta(u) \cap S_\alpha(x)\big\| + \big\|S_\delta(u) \cap S_\beta(x)\big\| + \big\|S_\delta(u) \cap S_\gamma(x)\big\| \\
&\leq 1 + 5 + 5 + 4 \\
&= 15,
\end{aligned}$$

which is impossible.

The proof is complete. $\square$

Theorem 16. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distince colors α, β, γ, and δ. Suppose that $u, v \in V$ with $S_\delta(u) = \{x_0, \ldots, x_{15}\}$ and $S_\delta(v) = \{y_0, \ldots, y_{15}\}$ with $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Suppose further that $\|V\| = 62$. If $\|S_\alpha(a)\| = 16$, then $\|S_\alpha(b)\| = \|S_\alpha(e)\| = 15$. If $\|S_\beta(a)\| = 16$, then $\|S_\beta(d)\| = \|S_\beta(c)\| = 15$.*

Proof. First, note that the hypotheses of the theorem are preserved by the following symmetry Θ:

$$\begin{gathered}
u \longmapsto u \\
v \longmapsto v \\
a \longmapsto a \\
b \longmapsto c \longmapsto e \longmapsto d \longmapsto b \\
x_0 \longmapsto x_0 \\
x_3 \longmapsto x_8 \longmapsto x_3 \\
x_1 \longmapsto x_7 \longmapsto x_5 \longmapsto x_9 \longmapsto x_1 \\
x_2 \longmapsto x_{10} \longmapsto x_4 \longmapsto x_6 \longmapsto x_2 \\
y_0 \longmapsto y_0 \\
y_3 \longmapsto y_8 \longmapsto y_3 \\
y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1 \\
y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2
\end{gathered}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{gathered}
\alpha \longmapsto \beta \longmapsto \alpha \\
\gamma \longmapsto \gamma \\
\delta \longmapsto \delta
\end{gathered}$$

First, we show that if $\|S_\alpha(a)\| = 16$, then $\|S_\alpha(b)\| = 15$. Suppose not. That is, suppose that $\|S_\alpha(a)\| = 15$ but $\|S_\alpha(b)\| \neq 16$. By Theorem 13, we see that $\|S_\gamma(b)\| \leq 14$, so that, by Proposition 1, we must have $\|S_\alpha(b)\| \geq 15$, so that, in fact,

$$\|S_\alpha(b)\| = \|S_\alpha(a)\| = 16.$$

Applying Theorem 13 once again, this time to a and b, and using the fact that $x_5, y_5, d \in S_\alpha(a) \cap S_\alpha(b)$ are pairwise distinct elements, we see that

$$\|S_\alpha(a) \cap S_\alpha(b)\| = 5.$$

Furthermore, note that $x_5 \overset{\delta}{\longrightarrow} y_5$, since $x_5, y_5 \in S_\alpha(a) \cap S_\beta(c) \cap S_\gamma(d)$. Note also that $d \overset{\gamma}{\longrightarrow} x_5, x_6$. Thus, by Theorem 13, applied to a and b, we see that

$$\|S_\beta(d)\| \leq 14.$$

But by Theorem 13, applied to u and v, we see that

$$\|S_\gamma(d)\| \leq 14,$$

which is impossible, by Proposition 1.

Thus, we have shown that

$$\text{if } \|S_\alpha(a)\| = 16, \text{ then } \|S_\alpha(b)\| = 15.$$

Repeated applications of the symmetry Θ give

$$\text{if } \|S_\beta(a)\| = 16, \text{ then } \|S_\beta(c)\| = 15,$$

$$\text{if } \|S_\alpha(a)\| = 16, \text{ then } \|S_\alpha(e)\| = 15,$$

and

$$\text{if } \|S_\beta(a)\| = 16, \text{ then } \|S_\beta(d)\| = 15.$$

The proof is complete. □

Theorem 17. *Let V be the vertex set of a complete graph with a good edge coloring, colored with four colors. Then there exists some color α and vertices $u, v \in V$, such that $\|S_\alpha(u)\| = \|S_\alpha(v)\| = 16$ and $\|S_\alpha(u) \cap S_\alpha(v)\| = 5$ with both $S_\alpha(u)$ and $S_\alpha(v)$ twisted.*

Proof. By Proposition 1, for each $z \in V$ there exists some color η such that $\|S_\eta(z)\| = 16$. Thus, we have

$$\big\|\{\, (z, \eta) \bigm| \|S_\eta(z)\| = 16 \,\}\big\| \geq 62.$$

Thus, there must exist some color δ such that

$$\big\|\{\, z \bigm| \|S_\delta(z)\| = 16 \,\}\big\| \geq 16.$$

Let $z_0, z_1, z_2, z_3, z_4, z_5 \in V$ be pairwise distinct vertices such that $\|S_\delta(z_i)\| = 16$ for all $i \in \{0, 1, 2, 3, 4, 5\}$.

First, we show that $\|S_\delta(z_i) \cap S_\delta(z_j)\| = 5$ for some $i, j \in \{0, 1, 2, 3, 4, 5\}$. Suppose not. Then, by Theorem 13, we see that $\|S_\delta(z_i) \cap S_\delta(z_j)\| \leq 2$ for any distinct $i, j \in \{0, 1, 2, 3, 4, 5\}$. Thus, we have

$$\begin{aligned} 62 &= \big\|V\big\| \\ &\geq \big\|S_\delta(z_0) \cup S_\delta(z_1) \cup S_\delta(z_2) \cup S_\delta(z_3) \cup S_\delta(z_4) \cup S_\delta(z_5)\big\| \\ &\geq 16 + 14 + 12 + 10 + 8 + 6 \\ &= 66, \end{aligned}$$

which is impossible. Thus, we see that there exist some $i, j \in \{0, 1, 2, 3, 4, 5\}$, such that $\|S_\delta(z_i) \cap S_\delta(z_j)\| = 5$. Let $x = z_i$ and $y = z_j$.

Thus, we have $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$ and $\|S_\delta(x) \cap S_\delta(y)\| = 5$. By Theorem 13, applied to x and y, we see that there exist colors α, β, and γ, such that $\|S_\gamma(w)\| \leq 14$ for all $w \in S_\delta(x) \cap S_\delta(y)$ and that all of the edges in $S_\delta(u) \cap S_\delta(v)$ are colored with the colors α and β.

By Proposition 1, we see that for any $w \in S_\delta(u) \cap S_\delta(v)$, we have either $\|S_\alpha(w)\| = 16$ or $\|S_\beta(w)\| = 16$ (or both). But $\|S_\delta(x) \cap S_\delta(y)\| = 5$, so that there must exist pairwise distinct $u, v, w \in S_\delta(x) \cap S_\delta(y)$ such that either $\|S_\alpha(u)\| = \|S_\alpha(v)\| = \|S_\alpha(w)\| = 16$ or $\|S_\beta(u)\| = \|S_\beta(v)\| = \|S_\beta(w)\| = 16$. Without loss of generality, we may suppose that

$$\|S_\alpha(u)\| = \|S_\alpha(v)\| = \|S_\alpha(w)\| = 16.$$

Since $\{u,v,w\} \subseteq S_\delta(x) \cap S_\delta(y)$, we see that all of the edges in $\{u,v,w\}$ must be colored with the colors α and β. Since all three edges in $\{u,v,w\}$ cannot be the same color, at least one such edge must be of color β. Without loss of generality, we may suppose that

$$u \stackrel{\beta}{\to} \!\!\!-\!\!\!-\, v.$$

By Theorem 1, applied to x and y, we see that there exist $w_0, w_1, w_2 \in S_\delta(x) \cap S_\delta(y)$ such that

$$P_{\beta,\alpha}(u, w_0, w_1, w_2, v).$$

Note that $w_1 \stackrel{\alpha}{\to} \!\!\!-\!\!\!-\, u, v$ and that w_1 is the only such element of $S_\delta(x) \cap S_\delta(y)$. Since $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$, we may apply Lemma 6(1), to see that there exists some $z \in S_\delta(x) \sim S_\delta(y)$ and some $z' \in S_\delta(y) \sim S_\delta(x)$ such that $w_1, z, z' \in S_\alpha(u) \cap S_\alpha(v)$. Clearly, the elements w_1, z, z' are pairwise distinct, so that, since $\|S_\alpha(u)\| = \|S_\alpha(v)\| = 16$, we see, by Theorem 13, applied to u and v, that $\|S_\alpha(u) \cap S_\alpha(v)\| = 5$. To see that $S_\alpha(u)$ and $S_\alpha(v)$ are twisted, we need only apply Theorem 13 to x and y, noting that $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$ and $\|S_\delta(x) \cap S_\delta(y)\| = 5$ and that the edges in $S_\delta(x) \cap S_\delta(y)$ are colored with the colors α and β.

The proof is complete. □

Theorem 18. $R(3,3,3,3;2) \leq 62$.

Proof. Suppose not. Then there exists a good edge coloring, using four colors, on a complete graph with vertex set V where $\|V\| = 62$. By Theorem 16, there exists some color δ and vertices $u, v \in V$ such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ and $\|S_\delta(u) \cap S_\delta(v)\| = 5$ with both $S_\delta(u)$ and $S_\delta(v)$ twisted.

By Theorem 13, there exist $x_0, \ldots, x_{15} \in S_\delta(u)$ and $y_0, \ldots, y_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{11, 12, 13, 14, 14\}$ and colors α, β, and γ, such that $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ with $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$, where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$.

First, note that the entire situation so far is preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto a$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_7 \longmapsto x_5 \longmapsto x_9 \longmapsto x_1$$
$$x_2 \longmapsto x_{10} \longmapsto x_4 \longmapsto x_6 \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1$$
$$y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

By Theorem 13, applied to u and v, we see that $\|S_\gamma(w)\| \leq 14$ for all $w \in \{a, b, c, d, e\}$.

Now, Suppose that $\|S_\alpha(a)\| = 16$. Then, by Theorem 15, applied to u and v, we have $\|S_\alpha(b)\| = \|S_\alpha(e)\| = 15$. But $\|S_\gamma(b)\| = \|S_\gamma(e)\| = 14$, so that, by Proposition 1, we have $\|S_\delta(b)\| = \|S_\delta(e)\| = \|S_\beta(b)\| = \|S_\beta(e)\| = 16$. Since $b, e, a \xrightarrow{\delta} \text{——} u, v$, we see, by Theorem 14, that $\|S_\delta(a)\| \leq 15$, so that, since $\|S_\delta(a)\| \leq 14$, we see, by Proposition 1, that $\|S_\beta(a)\| = 16$. Likewise, since $b, e, c \xrightarrow{\delta} \text{——} u, v$, we see, by Theorem 14, that $\|S_\delta(c)\| \leq 15$, so that, since $\|S_\delta(c)\| \leq 14$, we see, by Proposition 1, that $\|S_\beta(c)\| = 16$. But, by Theorem 15, applied to u and v, we cannot have $\|S_\beta(a)\| = \|S_\beta(c)\| = 16$, thus giving a contradiction.

Thus, we have shown that

$$\|S_\alpha(a)\| \leq 15.$$

An application of the symmetry to this gives

$$\|S_\beta(a)\| \leq 15.$$

Since $\|S_\gamma(a)\| \leq 14$, we get a contradiction, by Proposition 1.

The theorem is proved. ☐

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