Finite symmetric integral relation algebras with no 3-cycles

Roger D. Maddux
Department of Mathematics
396 Carver Hall
Iowa State University
Ames, Iowa 50011
U.S.A.
maddux@iastate.edu

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relation algebra

$$\mathfrak{A} = \langle A, +, ^{-}, ;, ^{\smile}, 1\text{'} \rangle$$

R_1	$x + y = y + x,$	+-commutativity
R_2	$x + (y + z) = (x + y) + z,$	+-associativity
R_3	$\overline{\overline{x} + \overline{y}} + \overline{\overline{x} + y} = x,$	Huntington's axiom
R_4	$x;(y;z) = (x;y);z,$	;-associativity
R_5	$(x + y);z = x;z + y;z,$	right ;-distributivity
R_6	$x;1\text{'} = x,$	right identity law
R_7	$\breve{\breve{x}} = x,$	$^{\smile}$-involution
R_8	$(x + y)^{\smile} = \breve{x} + \breve{y},$	$^{\smile}$-distributivity
R_9	$(x;y)^{\smile} = \breve{y};\breve{x},$	$^{\smile}$-involutive distributivity
R_{10}	$\breve{x};\overline{x;y} + \overline{y} = \overline{y}.$	Tarski/De Morgan axiom

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BA = class of Boolean algebras.

Boolean part of $\mathfrak{A} = \langle A, +, ^{-} \rangle \in \mathsf{BA}$ by R_1–R_3 (Huntington 1933)

RA = class of relation algebras

NA = class of **nonassociative relation algebras** (RA axioms except R_4)

$1' :=$	**identity element**
$x \cdot y := \overline{\overline{x} + \overline{y}}$	Boolean product
$x - y := \overline{\overline{x} + y}$	Boolean difference
$0 := \overline{1' + \overline{1'}}$	**zero element**
$0' := \overline{1'}$	**diversity element**
$1 := 1' + \overline{1'}$	**unit element**

$\mathsf{NA} \models$ **cycle law** (can replace R_5, R_7–R_{10})

$$\begin{aligned} \breve{x};z \cdot y = 0 \Leftrightarrow x;y \cdot z = 0 \Leftrightarrow z;\breve{y} \cdot x = 0 \\ \Leftrightarrow y;\breve{z} \cdot \breve{x} = 0 \Leftrightarrow \breve{y};\breve{x} \cdot \breve{z} = 0 \Leftrightarrow \breve{z};x \cdot \breve{y} = 0 \end{aligned}$$

$\mathsf{NA} \models$

$$\breve{1'} = 1' \quad \breve{0'} = 0'$$

$$1';x = x$$

$$0;x = x;0 = 0$$

$$1;1 = 1$$

$\mathfrak{A}$ is **symmetric** iff $\mathfrak{A} \models \breve{x} = x$

Symmetric implies **commutative**: $\mathfrak{A} \models x;y = y;x$

WA = class of **weakly associative relation algebras**

WA axioms = R_1–R_3,R_5–R_{10} plus **weak associative law**

$$((x \cdot 1');1);1 = (x \cdot 1');(1;1)$$

SA = class of **semiassociative relation algebras**

SA axioms = R_1–R_3,R_5–R_{10} plus **semiassociative law**

$$(x;1);1 = x;(1;1)$$

$$\mathsf{NA} \supset \mathsf{WA} \supset \mathsf{SA} \supset \mathsf{RA}$$

$At\mathfrak{A}$ = set of atoms of $\mathfrak{A}$

identity atom $x \in At\mathfrak{A}$: $x \leq 1'$

diversity atom $x \in At\mathfrak{A}$: $x \leq 0'$

integral: $0 \neq 1$ ($\mathfrak{A}$ is nontrivial) and $x;y = 0$ implies $x = 0$ or $y = 0$

$1' \in At\mathfrak{A}$ implies $\mathfrak{A}$ is integral
(converse holds for $\mathfrak{A} \in \mathsf{SA}$, fails for some $\mathfrak{A} \in \mathsf{WA}$)

$$\begin{aligned}
Sb\,(U) &:= \text{set of subsets of } U\\
\mathfrak{Bl}\,(U) &:= \langle Sb\,(U)\,, \cup, {}^{-} \rangle = \textbf{Boolean algebra of subsets of } U\\
X \cup Y &:= \text{ union of } X, Y \subseteq U\\
\overline{X} &:= U \sim X = \text{complement of } X \text{ with respect to } U
\end{aligned}$$

For an equivalence relation E

$$\begin{aligned}
\mathfrak{Sb}\,(E) &:= \left\langle Sb\,(E)\,, \cup, {}^{-}, |, {}^{-1}, \mathsf{Id} \cap E \right\rangle\\
R|S &:= \{\langle a, c\rangle : \exists_b(\langle a, b\rangle \in R,\ \langle b, c\rangle \in S)\}\\
&\quad = \text{relative product of } R, S \subseteq U^2\\
R^{-1} &:= \{\langle b, a\rangle : \langle a, b\rangle \in R\} = \text{converse of } R \subseteq U^2\\
\mathsf{Id} \cap E &:= \text{identity relation on the field } Fd\,(E) = \{x : \exists_y(xEy)\} \text{ of } E\\
\mathsf{Id} &:= \text{class of pairs of sets of the form } \langle x, x\rangle\\
\mathfrak{Sb}\,(E) &:= \textbf{relation algebra of subrelations of } E\\
\mathfrak{Sb}\,(E) &\text{ is an } \textbf{equivalence relation algebra}\\
Sb\,\left(U^2\right) &:= \textbf{set of binary relations on } U\\
\mathfrak{Re}\,(U) &:= \mathfrak{Sb}\,\left(U^2\right) = \textbf{ square relation algebra on } U
\end{aligned}$$

Every square relation algebra is an equivalence relation algebra
(but not conversely)

$\mathfrak{A}$ is a **proper relation algebra** iff
$\exists_E(E$ is an equivalence relation and $\mathfrak{A} \subseteq \mathfrak{Sb}\,(E))$

$\mathfrak{A}$ is a **representable relation algebra** if it is isomorphic to a proper relation algebra

RRA is the class of **representable relation algebras**

ρ is a **representation of $\mathfrak{A}$ over E** (and the field of E is the **base set** of ρ) if E is an equivalence relation and ρ is an embedding of $\mathfrak{A}$ into $\mathfrak{Sb}\,(E)$

$\mathfrak{A} \in \mathsf{RRA}$ iff there is a representation of $\mathfrak{A}$ over some equivalence relation

ρ is a **square representation of $\mathfrak{A}$ on U**
(and U is the **base set** of ρ) if ρ is a representation of $\mathfrak{A}$ over U^2

fRRA = class of **finitely representable relation algebras**
(algebras in RRA with a representation with a finite base set)

If an RRA has a representation with a finite base set, then it also has a representation with an infinite base set (not necessarily a *square* representation)

$\mathsf{RRA} = \mathbf{S}\mathsf{RRA} = \mathbf{P}\mathsf{RRA}$ (easy)

Tarski 1955: $\mathsf{RRA} = \mathbf{H}\mathsf{RRA}$

RRA has an equational axiomatization

Monk 1964: RRA has no finite axiomatization

Jónsson 1991: RRA has no equational basis with only finitely many variables

If ρ is a representation of $\mathfrak{A}$ over an equivalence relation E then

$$\rho(a+b) = \rho(a) \cup \rho(b) \tag{1}$$

$$\rho(\overline{a}) = E \sim \rho(a) \tag{2}$$

$$\rho(a \cdot b) = \rho(a) \cap \rho(b) \tag{3}$$

$$\rho(0) = \emptyset \tag{4}$$

$$\rho(a;b) = \rho(a) | \rho(b) \tag{5}$$

$$\rho(\breve{a}) = (\rho(a))^{-1} \tag{6}$$

$$\rho(1') = \mathsf{Id} \cap E \tag{7}$$

Weak representation (Jónsson 1959): drop (1) and (2)

ρ is a **weak representation** of $\mathfrak{A}$ **over** E if (3)–(7)

$\mathfrak{A} \in \mathsf{NA}$ is **weakly representable** if $\mathfrak{A}$ has a weak representation over some equivalence relation

wRRA = class of algebras in NA that have a weak representation

$\mathsf{RRA} \subseteq \mathsf{wRRA}$

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n**-by-**n **matrix** ($3 \leq n \leq \omega$) of $\mathfrak{A} \in \mathsf{NA}$: a function mapping n^2 into universe A of $\mathfrak{A}$

$B_n\mathfrak{A}$:= set of those n-by-n matrices of *atoms* of $\mathfrak{A}$ such that

$$\begin{aligned} &a_{ii} \leq 1\text{'} \\ &\breve{a}_{ij} = a_{ji} \\ &a_{ik} \leq a_{ij} ; a_{jk} \end{aligned}$$

N is an n**-dimensional relational basis for** $\mathfrak{A} \in \mathsf{NA}$ if

1. $\emptyset \neq N \subseteq B_n\mathfrak{A}$,
2. for every atom $x \in At\mathfrak{A}$ there is some $a \in N$ such that $a_{01} = x$,
3. if $a \in N$, $i, j, k < n$, $i, j \neq k$, $x, y \in At\mathfrak{A}$, and $a_{ij} \leq x ; y$, then for some $b \in N$, $a_{lm} = b_{lm}$ whenever $i \neq l, m < n$, $b_{ik} = x$, and $b_{kj} = y$

$B_n\mathfrak{Re}\,(U)$ is a relational basis for $\mathfrak{Re}\,(U)$

$\mathfrak{A} \in \mathsf{NA}$ is a **relation algebra of dimension** n if $\mathfrak{A}$ is a subalgebra of a complete atomic NA that has an n-dimensional relational basis

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RA_n is the **class of relation algebras of dimension** n

$$\mathsf{SA} = \mathsf{RA}_3 \supset \mathsf{RA} = \mathsf{RA}_4 \supset \mathsf{RA}_5 \supset \cdots \supset \mathsf{RA}_\omega = \mathsf{RRA}$$

$$t \cdot (u \cdot v ; w) ; (x \cdot y ; z) \leq v ; \big((\breve{v} ; t \cdot w ; x) ; \breve{z} \cdot w ; y \cdot \breve{v} ; (t ; \breve{z} \cdot u ; y) \big) ; z \qquad \text{(M)}$$

(M) comes from Jónsson 1959, Lyndon 1950

$$\mathsf{RA}_5 \models \text{(M)} \qquad \mathsf{wRRA} \models \text{(M)}$$

$$\begin{aligned} Cy(\mathfrak{A}) &:= \{\langle x,y,z\rangle : x,y,z \in At\mathfrak{A},\ x;y \geq z\} \\ &= \text{the } \textbf{cycle structure} \text{ of } \mathfrak{A} \\ [x,y,z] &:= \{\langle x,y,z\rangle, \langle \breve{x},z,y\rangle, \langle y,\breve{z},\breve{x}\rangle, \langle \breve{y},\breve{x},\breve{z}\rangle, \langle \breve{z},x,\breve{y}\rangle, \langle z,\breve{y},x\rangle\} \\ & \quad \text{is a } \textbf{cycle} \end{aligned}$$

The cycle structure of $\mathfrak{A}$ is a disjoint union of cycles (by the cycle law)

$[x,y,z]$ is a **forbidden cycle of $\mathfrak{A}$** if $[x,y,z] \cap Cy(\mathfrak{A}) = \emptyset$

$[x,y,z]$ is a **cycle of $\mathfrak{A}$** if $[x,y,z] \subseteq Cy(\mathfrak{A})$

$[x,y,z]$ is an **identity cycle** if one (or, equivalently, all) of its triples contains an identity atom

$[x,y,z]$ is a **diversity cycle** if all of the elements in its triples are diversity atoms

Assume $\mathfrak{A} \in \mathsf{NA}$ is symmetric and 1' $\in At\mathfrak{A}$

$[x,y,z]$ is a **3-cycle** of $\mathfrak{A}$ iff $[x,y,z] \subseteq Cy(\mathfrak{A})$ and $|\{x,y,z\}| = 3$

3-cycle $[a,b,c]$ 2-cycle $[a,b,b]$ 1-cycle $[a,a,a]$

$\mathfrak{A}$ has no 3-cycles if every 3-cycle is forbidden

T is a ternary relation and $U := \{x : \exists_y \exists_z (Txyz \text{ or } Tyxz \text{ or } Tyzx)\}$

Define

$$\begin{aligned} X\,;Y &:= \{c : \exists_x \exists_y (x \in X, y \in Y, Txyc)\}, \quad X, Y \subseteq U \\ S &:= \{\langle a, b\rangle : a, b \in U, \forall_x \forall_y ((Taxy \Leftrightarrow Tbyx),\ (Txay \Leftrightarrow Tybx))\} \subseteq U^2 \\ \breve{X} &:= \{b : \exists_x (Sxb, x \in X)\}, \quad X \subseteq U \\ I &:= \{a : a \in U, \forall_x \forall_y ((Taxy \text{ or } Txay) \Rightarrow x = y)\} \subseteq U \end{aligned}$$

The **complex algebra** of T is $\mathfrak{Cm}\,(T) := \langle Sb\,(U)\,, \cup, ^{-}, ;, ^{\smile}, I\rangle$

Consider statements

$$(\mathsf{NA}) \quad \forall_a (a \in U \Rightarrow \exists_b Sab) \tag{C}$$

$$(\mathsf{NA}) \quad \forall_a (a \in U \Rightarrow \exists_i (i \in I,\ Tiaa)) \tag{I}$$

$$(\mathsf{RA}) \quad \forall_x \forall_y \forall_z \forall_a \forall_b (Txyz,\ Tzab \Rightarrow \exists_c (Txcb,\ Tyac)) \tag{A}$$

$$(\mathsf{SA}) \quad \forall_x \forall_y \forall_z \forall_a \forall_b (Txyz,\ Tzab \Rightarrow \exists_c Txcb) \tag{S}$$

$$(\mathsf{WA}) \quad \forall_x \forall_y \forall_z \forall_a \forall_b (Txyz,\ Tzab,\ Ix \Rightarrow \exists_c Txcb) \tag{W}$$

$$(\mathsf{WA}) \quad \forall_x \forall_z \forall_a \forall_b (Txzz,\ Tzab,\ Ix \Rightarrow Txbb) \tag{W$'$}$$

If (C)(I) then S is an involution ($S : U \to U$, $S(S(x)) = x$)

- $\mathfrak{Cm}\,(T) \in \mathsf{NA} \Leftrightarrow$ (C)(I)
- $\mathfrak{Cm}\,(T) \in \mathsf{RA} \Leftrightarrow$ (C)(I)(A)
- $\mathfrak{Cm}\,(T) \in \mathsf{SA} \Leftrightarrow$ (C)(I)(S)
- $\mathfrak{Cm}\,(T) \in \mathsf{WA} \Leftrightarrow$ (C)(I)(W)
- $\mathfrak{Cm}\,(T) \in \mathsf{WA} \Leftrightarrow$ (C)(I)(W$'$)

If $I = \{1\text{'}\}$ then (I) becomes (I$'$) $\forall_a (a \in U \Rightarrow T1\text{'}aa)$

Every square relation algebra on a set U is a complex algebra

$$\mathfrak{Re}\,(U) = \mathfrak{Cm}\,(\{\langle\langle a,b\rangle\,,\langle b,c\rangle\,,\langle a,c\rangle\rangle : a,b,c \in U\})$$

Jónsson-Tarski 1952: Every RA is isomorphic to a subalgebra of a complex algebra of a ternary relation

$$\mathsf{RA} = \{\mathfrak{A} : \mathfrak{A} \in \mathbf{S}\{\mathfrak{Cm}\,(T)\}, T \models \text{(C)(I)(A)}\}$$

If $\mathfrak{G} = \langle G, \circ\rangle$ is a group and

$$\circ = \{\langle x,y,z\rangle : x,y,z \in G,\ x \circ y = z\}$$

then $\mathfrak{Cm}\,(\mathfrak{G}) := \mathfrak{Cm}\,(\circ)$

GRA := algebras isomorphic to a subalgebra of $\mathfrak{Cm}\,(\mathfrak{G})$ for some group $\mathfrak{G}$

GRA := **group relation algebras**

Every group relation algebra has a square representation over itself

$$\mathsf{GRA} \subseteq \mathsf{RRA}$$

Assume $\mathfrak{A} \in \mathsf{RA}$, $\mathfrak{A}$ is symmetric, atomic, integral, and has no 3-cycles

Define binary relations $\rightarrow$, $\Rightarrow$, and $\Leftrightarrow$, on diversity atoms $x, y \in At\mathfrak{A}$

- $x \rightarrow y$ iff $x \neq y$ and $x \leq y ; y$
- $x \Rightarrow y$ iff $x = y$ or $x \rightarrow y$
- $x \Leftrightarrow y$ iff $x \Rightarrow y$ and $y \Rightarrow x$

$$\begin{aligned} [x] &= \{y : 0\text{'} \geq y \in \mathrm{At}\mathfrak{A},\ x \Leftrightarrow y\} \\ D &= \{[x] : 0\text{'} \geq x \in \mathrm{At}\mathfrak{A}\} \\ [\Rightarrow] &= \{\langle [x], [y] \rangle : 0\text{'} \geq x, y \in \mathrm{At}\mathfrak{A},\ x \Rightarrow y\} \end{aligned}$$

Then

- if $x \rightarrow y$, $y \rightarrow z$, and $x \neq z$, then $x \rightarrow z$
- $\Rightarrow$ is reflexive and transitive
- either $x \Rightarrow y$ or $y \Rightarrow x$
- $\Leftrightarrow$ is an equivalence relation
- $[\Rightarrow]$ is a linear ordering of D

Assume $\mathfrak{A} \in \mathsf{RA}$, $\mathfrak{A}$ is symmetric, atomic, integral, and has no 3-cycles

Assume $\mathfrak{A}$ is also finite and $n = |D|$

Choose representatives $a_1, \ldots, a_n \in At\mathfrak{A}$ from the equivalence classes in D so that

$$\begin{aligned}
&a_1 \longrightarrow a_2 \longrightarrow a_3 \longrightarrow \cdots \longrightarrow a_{n-2} \longrightarrow a_{n-1} \longrightarrow a_n \\
&[a_1] \cup \cdots \cup [a_n] = At\mathfrak{A} \sim \{1'\} \\
&[a_1][\Rightarrow][a_2][\Rightarrow][a_3][\Rightarrow] \ldots [\Rightarrow][a_{n-2}][\Rightarrow][a_{n-1}][\Rightarrow][a_n]
\end{aligned}$$

Define the **cycle parameters** of $\mathfrak{A}$

$$\begin{aligned}
s_i &:= \text{ number of atoms in } [a_i] \text{ that appear in a 1-cycle of } \mathfrak{A} \\
s_i &:= |[a_i] \cap \{a : a \leq a;a\}| \\
t_i &:= \text{ number of atoms in } [a_i] \text{ that do not appear in a 1-cycle of } \mathfrak{A} \\
t_i &:= |[a_i] \cap \{a : 0 = a \cdot a;a\}|
\end{aligned}$$

$$\mathrm{Cp}(\mathfrak{A}) := \begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix}$$

If $\mathfrak{A}$ has no diversity atoms, $\mathrm{Cp}(\mathfrak{A}) := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For finite symmetric integral relation algebras with no 3-cycles, the isomorphism type of $\mathfrak{A}$ is determined by $\mathrm{Cp}(\mathfrak{A})$

Almost any two sequences of nonnegative integers with the same length determine a finite symmetric integral relation algebra with no 3-cycles

- $\mathrm{Cp}(\mathfrak{A}) = \mathrm{Cp}(\mathfrak{B}) \Rightarrow \mathfrak{A} \cong \mathfrak{B}$
- If

$$\begin{array}{lll} n \in \omega \qquad s_1, \cdots, s_n \in \omega \qquad t_1, \cdots, t_n \in \omega \\ 0 < s_1 + t_1 \quad \cdots \quad 0 < s_n + t_n \end{array}$$

  then there is a finite symmetric integral relation algebra $\mathfrak{A}$ with no 3-cycles such that

$$\mathrm{Cp}(\mathfrak{A}) = \begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix}$$

Assume $\mathfrak{A}, \mathfrak{B} \in \mathsf{NA}$ are finite and $\{1'\} = At\mathfrak{A} \cap At\mathfrak{B}$

2-cycle product $\mathfrak{A}[\mathfrak{B}]$ of $\mathfrak{A}$ and $\mathfrak{B}$ is the complex algebra of

$$T := Cy(\mathfrak{A}) \cup Cy(\mathfrak{B}) \cup \{[a, b, b] : a \in At\mathfrak{A} \sim \{1'\}, b \in At\mathfrak{B} \sim \{1'\}\}$$
$$\mathfrak{A}[\mathfrak{B}] := \mathfrak{Cm}\,(T)$$

Assume also

$At\mathfrak{A} \sim \{1'\} \neq \emptyset \neq At\mathfrak{B} \sim \{1'\}$

$\mathfrak{A}$, $\mathfrak{B}$ are symmetric

$\mathfrak{A}$, $\mathfrak{B}$ have no 3-cycles

$$\mathrm{Cp}(\mathfrak{A}) = \begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix} \qquad \mathrm{Cp}(\mathfrak{B}) = \begin{pmatrix} s'_1 & \cdots & s'_n \\ t'_1 & \cdots & t'_n \end{pmatrix}$$

where $0 < s_1 + t_1, \ldots, 0 < s_n + t_n, 0 < s'_1 + t'_1, \ldots, 0 < s'_n + t'_n$

Then

$$\mathrm{Cp}(\mathfrak{A}[\mathfrak{B}]) = \begin{pmatrix} s_1 & \cdots & s_n & s'_1 & \cdots & s'_n \\ t_1 & \cdots & t_n & t'_1 & \cdots & t'_n \end{pmatrix}$$

Comer 1983: $\mathfrak{A}[\mathfrak{B}] \in \mathsf{RRA}$ iff $\mathfrak{A}, \mathfrak{B} \in \mathsf{RRA}$

If σ is a square representation of $\mathfrak{A}$ on U and
τ is a square representation of $\mathfrak{B}$ on V, then
there is a square representation φ of $\mathfrak{A}[\mathfrak{B}]$ on $U \times V$

Comer 1983: $\mathfrak{A}[\mathfrak{B}] \in \mathsf{GRA}$ iff $\mathfrak{A}, \mathfrak{B} \in \mathsf{GRA}$

If σ is an embedding of $\mathfrak{A}$ into $\mathfrak{Cm}\,(\mathfrak{G})$, and
τ is an embedding of $\mathfrak{B}$ into $\mathfrak{Cm}\,(\mathfrak{H})$, then
there is an embedding φ of $\mathfrak{A}[\mathfrak{B}]$ into $\mathfrak{Cm}\,(\mathfrak{G} \times \mathfrak{H})$

Assume $\mathrm{Cp}(\mathfrak{A}) = \begin{pmatrix} i \\ 0 \end{pmatrix}$

$\mathfrak{A}$ can be embedded in the complex algebra of an infinite group if $\mathrm{Cp}(\mathfrak{A})$ is one of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \ldots$

$\mathfrak{A} \subseteq \mathfrak{Cm}\,$(a finite group) iff $\mathrm{Cp}(\mathfrak{A})$ is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\mathfrak{A}$ has *no* representation on a finite set if $\mathrm{Cp}(\mathfrak{A})$ is one of $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \ldots$

Let V be the variety with the equational basis

$$\begin{aligned}
&\mathrm{R}_1\text{–}\mathrm{R}_{10}\\
&\breve{x} = x\\
&x ; y \leq x ; (x \cdot y) + x + y\\
&(x \cdot 1') ; 1 ; (y \cdot 1') = (x \cdot y \cdot 1') ; 1 ; (x \cdot y \cdot 1')
\end{aligned}$$

Simple finite algebras in V are symmetric integral relation algebras with no 3-cycles

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Classifying finite algebras in V—

Let $\mathfrak{A}$ be a finite symmetric integral RA with no 3-cycles

Suppose $\mathrm{Cp}(\mathfrak{A}) = \begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix}$

- The following statements are equivalent
  - (a) $\mathfrak{A} \in \mathsf{GRA}$
  - (b) $\mathfrak{A} \in \mathsf{RRA}$
  - (c) $\mathfrak{A} \in \mathsf{wRRA}$
  - (d) $\mathfrak{A} \in \mathsf{RA}_5$
  - (e) $\mathfrak{A}$ satisfies (M)
  - (f) for every $i \in \{1, \ldots, n\}$, if $t_i > 0$ then $s_i + t_i \leq 2$
- If (a)–(f) hold, then the following statements are equivalent
  - (g) $\mathfrak{A}$ is representable over a finite set
  - (h) for every $i \in \{1, \ldots, n\}$, if $s_i = 0$ then $t_i \leq 2$

There is an algorithm for classifying each finite $\mathfrak{A} \in \mathsf{V}$ as either in fRRA, in $\mathsf{GRA} \sim \mathsf{fRRA}$, or in $\mathsf{RA} \sim \mathsf{RA}_5$

Algorithm for classifying finite $\mathfrak{A} \in \mathsf{V}$—

Assume

$$\mathrm{Cp}(\mathfrak{A}) = \begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix}$$

Classify columns in $\mathrm{Cp}(\mathfrak{A})$:

$$\begin{array}{cccccc}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} f & \begin{pmatrix} 1 \\ 0 \end{pmatrix} f & \begin{pmatrix} 2 \\ 0 \end{pmatrix} f & \begin{pmatrix} 3 \\ 0 \end{pmatrix} \infty & \begin{pmatrix} 4 \\ 0 \end{pmatrix} \infty & \ldots \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} f & \begin{pmatrix} 1 \\ 1 \end{pmatrix} f & \begin{pmatrix} 2 \\ 1 \end{pmatrix} M & \begin{pmatrix} 3 \\ 1 \end{pmatrix} M & \begin{pmatrix} 4 \\ 1 \end{pmatrix} M & \ldots \\
\begin{pmatrix} 0 \\ 2 \end{pmatrix} f & \begin{pmatrix} 1 \\ 2 \end{pmatrix} M & \begin{pmatrix} 2 \\ 2 \end{pmatrix} M & \begin{pmatrix} 3 \\ 2 \end{pmatrix} M & \begin{pmatrix} 4 \\ 2 \end{pmatrix} M & \ldots \\
\begin{pmatrix} 0 \\ 3 \end{pmatrix} M & \begin{pmatrix} 1 \\ 3 \end{pmatrix} M & \begin{pmatrix} 2 \\ 3 \end{pmatrix} M & \begin{pmatrix} 3 \\ 3 \end{pmatrix} M & \begin{pmatrix} 4 \\ 3 \end{pmatrix} M & \ldots
\end{array}$$

If an M-column appears in $\mathrm{Cp}(\mathfrak{A})$, then
(M) fails and $\mathfrak{A} \in \mathsf{RA} \sim (\mathsf{GRA} \cup \mathsf{RRA} \cup \mathsf{wRRA} \cup \mathsf{RA}_5)$

If no M-column appears in $\mathrm{Cp}(\mathfrak{A})$ and an ∞ column appears in $\mathrm{Cp}(\mathfrak{A})$, then
$\mathfrak{A} \in \mathsf{GRA} \sim \mathsf{fRRA}$

If all columns in $\mathrm{Cp}(\mathfrak{A})$ are f-columns, then
$\mathfrak{A} \in \mathsf{GRA} \cap \mathsf{fRRA}$

One relation algebra with a single atom: $\mathsf{1}_1$

$$\mathrm{Cp}(\mathsf{1}_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathsf{1}_1 \cong \mathfrak{Cm}\,(\mathbb{Z}_1)$$

Two relation algebras with two atoms: $\mathsf{1}_2, \mathsf{2}_2 \in \mathsf{GRA}$

$\mathsf{1}_2$	1’	0’
1’	1’	0’
0’	0’	1’

$$\mathrm{Cp}(\mathsf{1}_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{1}_2 \cong \mathfrak{Cm}\,(\mathbb{Z}_2)$$

$\mathsf{2}_2$	1’	0’
1’	1’	0’
0’	0’	1’0’

$$\mathrm{Cp}(\mathsf{2}_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathsf{2}_2 \subseteq \mathfrak{Cm}\,(\mathbb{Z}_n) \quad \text{for all } n \geq 3$$

$\mathsf{1}_2$ has a square representation on a 2-element set, but it does not have a square representation on a set of any other cardinality

$\mathsf{2}_2$ has square representations on sets of every cardinality ≥ 3

Seven symmetric integral relation algebras with three atoms: $\mathsf{1}_7$–$\mathsf{7}_7$

Cycles of $\mathsf{1}_7$–$\mathsf{7}_7$ ($aaa = [a, a, a]$)

	$1'1'1'$	$1'aa$	$1'bb$	aaa	bbb	abb	baa
$\mathsf{1}_7$	$1'1'1'$	$1'aa$	$1'bb$	$\cdots$	$\cdots$	abb	$\cdots$
$\mathsf{2}_7$	$1'1'1'$	$1'aa$	$1'bb$	aaa	$\cdots$	abb	$\cdots$
$\mathsf{3}_7$	$1'1'1'$	$1'aa$	$1'bb$	$\cdots$	bbb	abb	$\cdots$
$\mathsf{4}_7$	$1'1'1'$	$1'aa$	$1'bb$	aaa	bbb	abb	$\cdots$
$\mathsf{5}_7$	$1'1'1'$	$1'aa$	$1'bb$	$\cdots$	$\cdots$	abb	baa
$\mathsf{6}_7$	$1'1'1'$	$1'aa$	$1'bb$	aaa	$\cdots$	abb	baa
$\mathsf{7}_7$	$1'1'1'$	$1'aa$	$1'bb$	aaa	bbb	abb	baa

Multiplication tables for the atoms of the seven algebras $\mathsf{1}_7$–$\mathsf{7}_7$

$\mathsf{1}_7$	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'$	b
b	b	b	$1'a$

$\mathsf{2}_7$	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'a$	b
b	b	b	$1'a$

$\mathsf{3}_7$	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'$	b
b	b	b	$1'ab$

$\mathsf{4}_7$	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'a$	b
b	b	b	$1'ab$

$\mathsf{5}_7$	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'b$	ab
b	b	ab	$1'a$

$\mathsf{6}_7$	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'ab$	ab
b	b	ab	$1'a$

$\mathsf{7}_7$	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'ab$	ab
b	b	ab	$1'ab$

Cycle parameters of $\mathsf{1}_7$–$\mathsf{7}_7$

$$\mathrm{Cp}(\mathsf{1}_7) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathrm{Cp}(\mathsf{2}_7) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathrm{Cp}(\mathsf{3}_7) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{Cp}(\mathsf{4}_7) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{5}_7) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \mathrm{Cp}(\mathsf{6}_7) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathrm{Cp}(\mathsf{7}_7) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
&\mathsf{1}_7 \subseteq \mathfrak{Cm}\left(\mathbb{Z}_2^2\right) \\
&\mathsf{2}_7 \subseteq \mathfrak{Cm}\left(\mathbb{Z}_3 \times \mathbb{Z}_2\right) \cong \mathfrak{Cm}\left(\mathbb{Z}_6\right) \\
&\mathsf{3}_7 \subseteq \mathfrak{Cm}\left(\mathbb{Z}_2 \times \mathbb{Z}_3\right) \cong \mathfrak{Cm}\left(\mathbb{Z}_6\right) \\
&\mathsf{4}_7 \subseteq \mathfrak{Cm}\left(\mathbb{Z}_3^2\right) \\
&\mathsf{5}_7 \subseteq \mathfrak{Cm}\left(\mathbb{Z}_5\right) \\
&\mathsf{6}_7 \subseteq \mathfrak{Cm}\left(\mathbb{Z}_8\right) \\
&\mathsf{7}_7 \subseteq \mathfrak{Cm}\left(\mathbb{Z}_3^2\right)
\end{aligned}$$

Out of 65 symmetric integral relation algebras that have four atoms, the ones which have no 3-cycles are 1_{65}–24_{65}

	aaa	bbb	ccc	abb	baa	acc	caa	bcc	cbb
1_{65}	$\cdots$	$\cdots$	$\cdots$	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
2_{65}	aaa	$\cdots$	$\cdots$	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
3_{65}	$\cdots$	bbb	$\cdots$	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
4_{65}	aaa	bbb	$\cdots$	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
5_{65}	$\cdots$	$\cdots$	ccc	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
6_{65}	aaa	$\cdots$	ccc	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
7_{65}	$\cdots$	bbb	ccc	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
8_{65}	aaa	bbb	ccc	abb	$\cdots$	acc	$\cdots$	bcc	$\cdots$
9_{65}	$\cdots$	$\cdots$	$\cdots$	abb	baa	acc	$\cdots$	bcc	$\cdots$
10_{65}	aaa	$\cdots$	$\cdots$	abb	baa	acc	$\cdots$	bcc	$\cdots$
11_{65}	aaa	bbb	$\cdots$	abb	baa	acc	$\cdots$	bcc	$\cdots$
12_{65}	$\cdots$	$\cdots$	ccc	abb	baa	acc	$\cdots$	bcc	$\cdots$
13_{65}	aaa	$\cdots$	ccc	abb	baa	acc	$\cdots$	bcc	$\cdots$
14_{65}	aaa	bbb	ccc	abb	baa	acc	$\cdots$	bcc	$\cdots$
15_{65}	$\cdots$	$\cdots$	$\cdots$	$\cdots$	baa	acc	caa	bcc	$\cdots$
16_{65}	aaa	$\cdots$	$\cdots$	$\cdots$	baa	acc	caa	bcc	$\cdots$
17_{65}	$\cdots$	bbb	$\cdots$	$\cdots$	baa	acc	caa	bcc	$\cdots$
18_{65}	aaa	bbb	$\cdots$	$\cdots$	baa	acc	caa	bcc	$\cdots$
19_{65}	aaa	$\cdots$	ccc	$\cdots$	baa	acc	caa	bcc	$\cdots$
20_{65}	aaa	bbb	ccc	$\cdots$	baa	acc	caa	bcc	$\cdots$
21_{65}	$\cdots$	$\cdots$	$\cdots$	abb	baa	acc	caa	bcc	cbb
22_{65}	aaa	$\cdots$	$\cdots$	abb	baa	acc	caa	bcc	cbb
23_{65}	aaa	bbb	$\cdots$	abb	baa	acc	caa	bcc	cbb
24_{65}	aaa	bbb	ccc	abb	baa	acc	caa	bcc	cbb

Multiplication tables for the atoms of algebras 1_{65}–24_{65}

1_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’	b	c
b	b	b	1’a	c
c	c	c	c	1’ab

2_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’a	b	c
b	b	b	1’a	c
c	c	c	c	1’ab

3_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’	b	c
b	b	b	1’ab	c
c	c	c	c	1’ab

4_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’a	b	c
b	b	b	1’ab	c
c	c	c	c	1’ab

5_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’	b	c
b	b	b	1’a	c
c	c	c	c	1’abc

6_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’a	b	c
b	b	b	1’a	c
c	c	c	c	1’abc

7_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’	b	c
b	b	b	1’ab	c
c	c	c	c	1’abc

8_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’a	b	c
b	b	b	1’ab	c
c	c	c	c	1’abc

9_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’b	ab	c
b	b	ab	1’a	c
c	c	c	c	1’ab

10_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’ab	ab	c
b	b	ab	1’a	c
c	c	c	c	1’ab

11_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’ab	ab	c
b	b	ab	1’ab	c
c	c	c	c	1’ab

12_{65}	1’	a	b	c
1’	1’	a	b	c
a	a	1’b	ab	c
b	b	ab	1’a	c
c	c	c	c	1’abc

13_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'ab$	ab	c
b	b	ab	$1'a$	c
c	c	c	c	$1'abc$

14_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'ab$	ab	c
b	b	ab	$1'ab$	c
c	c	c	c	$1'abc$

15_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'bc$	a	ac
b	b	a	$1'$	c
c	c	ac	c	$1'ab$

16_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'abc$	a	ac
b	b	a	$1'$	c
c	c	ac	c	$1'ab$

17_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'bc$	a	ac
b	b	a	$1'b$	c
c	c	ac	c	$1'ab$

18_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'abc$	a	ac
b	b	a	$1'b$	c
c	c	ac	c	$1'ab$

19_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'abc$	a	ac
b	b	a	$1'$	c
c	c	ac	c	$1'abc$

20_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'abc$	a	ac
b	b	a	$1'b$	c
c	c	ac	c	$1'abc$

21_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'bc$	ab	ac
b	b	ab	$1'ac$	bc
c	c	ac	bc	$1'ab$

22_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'abc$	ab	ac
b	b	ab	$1'ac$	bc
c	c	ac	bc	$1'ab$

23_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'abc$	ab	ac
b	b	ab	$1'abc$	bc
c	c	ac	bc	$1'ab$

24_{65}	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1'abc$	ab	ac
b	b	ab	$1'abc$	bc
c	c	ac	bc	$1'abc$

$$\mathrm{Cp}(\mathsf{1}_{65}) = \begin{pmatrix} 0\,0\,0 \\ 1\,1\,1 \end{pmatrix} \mathrm{Cp}(\mathsf{2}_{65}) = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \end{pmatrix} \mathrm{Cp}(\mathsf{3}_{65}) = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,1 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{4}_{65}) = \begin{pmatrix} 1\,1\,0 \\ 0\,0\,1 \end{pmatrix} \mathrm{Cp}(\mathsf{5}_{65}) = \begin{pmatrix} 0\,0\,1 \\ 1\,1\,0 \end{pmatrix} \mathrm{Cp}(\mathsf{6}_{65}) = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,0 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{7}_{65}) = \begin{pmatrix} 0\,1\,1 \\ 1\,0\,0 \end{pmatrix} \mathrm{Cp}(\mathsf{8}_{65}) = \begin{pmatrix} 1\,1\,1 \\ 0\,0\,0 \end{pmatrix} \mathrm{Cp}(\mathsf{9}_{65}) = \begin{pmatrix} 0\,0 \\ 2\,1 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{10}_{65}) = \begin{pmatrix} 1\,0 \\ 1\,1 \end{pmatrix} \mathrm{Cp}(\mathsf{11}_{65}) = \begin{pmatrix} 2\,0 \\ 0\,1 \end{pmatrix} \mathrm{Cp}(\mathsf{12}_{65}) = \begin{pmatrix} 0\,1 \\ 2\,0 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{13}_{65}) = \begin{pmatrix} 1\,1 \\ 1\,0 \end{pmatrix} \mathrm{Cp}(\mathsf{14}_{65}) = \begin{pmatrix} 2\,1 \\ 0\,0 \end{pmatrix} \mathrm{Cp}(\mathsf{15}_{65}) = \begin{pmatrix} 0\,0 \\ 1\,2 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{16}_{65}) = \begin{pmatrix} 0\,1 \\ 1\,1 \end{pmatrix} \mathrm{Cp}(\mathsf{17}_{65}) = \begin{pmatrix} 1\,0 \\ 0\,2 \end{pmatrix} \mathrm{Cp}(\mathsf{18}_{65}) = \begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{19}_{65}) = \begin{pmatrix} 0\,2 \\ 1\,0 \end{pmatrix} \mathrm{Cp}(\mathsf{20}_{65}) = \begin{pmatrix} 1\,2 \\ 0\,0 \end{pmatrix} \mathrm{Cp}(\mathsf{21}_{65}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{22}_{65}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \mathrm{Cp}(\mathsf{23}_{65}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathrm{Cp}(\mathsf{24}_{65}) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathsf{1}_{65}, \cdots, \mathsf{20}_{65} &\in \mathsf{GRA} \cap \mathsf{fRRA} \\ \mathsf{21}_{65}, \mathsf{22}_{65}, \mathsf{23}_{65} &\not\models (\mathrm{M}) \\ \mathsf{21}_{65}, \mathsf{22}_{65}, \mathsf{23}_{65} &\in \mathsf{RA} \sim (\mathsf{GRA} \cup \mathsf{RRA} \cup \mathsf{wRRA} \cup \mathsf{RA}_5) \\ \mathsf{24}_{65} &\in \mathsf{GRA} \sim \mathsf{fRRA} \end{aligned}$$

Out of 3013 symmetric integral relation algebras with five atoms, the ones that have no 3-cycles are 1_{3013}–82_{3013}

	aaa	*bbb*	*ccc*	*ddd*	*abb*	*baa*	*acc*	*caa*	*add*	*daa*	*bcc*	*cbb*	*bdd*	*dbb*	*cdd*	*dcc*
1_{3013}	···	···	···	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
2_{3013}	*aaa*	···	···	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
3_{3013}	···	*bbb*	···	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
4_{3013}	*aaa*	*bbb*	···	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
5_{3013}	···	···	*ccc*	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
6_{3013}	*aaa*	···	*ccc*	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
7_{3013}	···	*bbb*	*ccc*	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
8_{3013}	*aaa*	*bbb*	*ccc*	···	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
9_{3013}	···	···	···	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
10_{3013}	*aaa*	···	···	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
11_{3013}	···	*bbb*	···	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
12_{3013}	*aaa*	*bbb*	···	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
13_{3013}	···	···	*ccc*	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
14_{3013}	*aaa*	···	*ccc*	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
15_{3013}	···	*bbb*	*ccc*	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
16_{3013}	*aaa*	*bbb*	*ccc*	*ddd*	*abb*	···	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
17_{3013}	···	···	···	···	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
18_{3013}	*aaa*	···	···	···	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
19_{3013}	*aaa*	*bbb*	···	···	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
20_{3013}	···	···	*ccc*	···	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
21_{3013}	*aaa*	···	*ccc*	···	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
22_{3013}	*aaa*	*bbb*	*ccc*	···	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
23_{3013}	···	···	···	*ddd*	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
24_{3013}	*aaa*	···	···	*ddd*	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
25_{3013}	*aaa*	*bbb*	···	*ddd*	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
26_{3013}	···	···	*ccc*	*ddd*	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
27_{3013}	*aaa*	···	*ccc*	*ddd*	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···
28_{3013}	*aaa*	*bbb*	*ccc*	*ddd*	*abb*	*baa*	*acc*	···	*add*	···	*bcc*	···	*bdd*	···	*cdd*	···

	aaa	bbb	ccc	ddd	abb	baa	acc	caa	add	daa	bcc	cbb	bdd	dbb	cdd	dcc
29_{3013}	···	···	···	···	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
30_{3013}	aaa	···	···	···	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
31_{3013}	···	bbb	···	···	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
32_{3013}	aaa	bbb	···	···	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
33_{3013}	aaa	···	ccc	···	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
34_{3013}	aaa	bbb	ccc	···	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
35_{3013}	···	···	···	ddd	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
36_{3013}	aaa	···	···	ddd	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
37_{3013}	···	bbb	···	ddd	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
38_{3013}	aaa	bbb	···	ddd	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
39_{3013}	aaa	···	ccc	ddd	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
40_{3013}	aaa	bbb	ccc	ddd	···	baa	acc	caa	add	···	bcc	···	bdd	···	cdd	···
41_{3013}	···	···	···	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
42_{3013}	aaa	···	···	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
43_{3013}	···	bbb	···	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
44_{3013}	aaa	bbb	···	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
45_{3013}	···	···	ccc	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
46_{3013}	aaa	···	ccc	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
47_{3013}	···	bbb	ccc	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
48_{3013}	aaa	bbb	ccc	···	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
49_{3013}	aaa	···	···	ddd	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
50_{3013}	aaa	bbb	···	ddd	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
51_{3013}	aaa	···	ccc	ddd	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
52_{3013}	aaa	bbb	ccc	ddd	···	baa	···	caa	add	daa	bcc	···	bdd	···	cdd	···
53_{3013}	···	···	···	···	abb	baa	acc	caa	add	···	bcc	cbb	bdd	···	cdd	···
54_{3013}	aaa	···	···	···	abb	baa	acc	caa	add	···	bcc	cbb	bdd	···	cdd	···
55_{3013}	aaa	bbb	···	···	abb	baa	acc	caa	add	···	bcc	cbb	bdd	···	cdd	···
56_{3013}	aaa	bbb	ccc	···	abb	baa	acc	caa	add	···	bcc	cbb	bdd	···	cdd	···
57_{3013}	···	···	···	ddd	abb	baa	acc	caa	add	···	bcc	cbb	bdd	···	cdd	···
58_{3013}	aaa	···	···	ddd	abb	baa	acc	caa	add	···	bcc	cbb	bdd	···	cdd	···
59_{3013}	aaa	bbb	···	ddd	abb	baa	acc	caa	add	···	bcc	cbb	bdd	···	cdd	···

	aaa	bbb	ccc	ddd	abb	baa	acc	caa	add	daa	bcc	cbb	bdd	dbb	cdd	dcc
60_{3013}	aaa	bbb	ccc	ddd	abb	baa	acc	caa	add	$\cdots$	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
61_{3013}	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
62_{3013}	aaa	$\cdots$	$\cdots$	$\cdots$	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
63_{3013}	$\cdots$	bbb	$\cdots$	$\cdots$	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
64_{3013}	aaa	bbb	$\cdots$	$\cdots$	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
65_{3013}	$\cdots$	bbb	ccc	$\cdots$	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
66_{3013}	aaa	bbb	ccc	$\cdots$	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
67_{3013}	aaa	$\cdots$	$\cdots$	ddd	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
68_{3013}	aaa	bbb	$\cdots$	ddd	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
69_{3013}	aaa	bbb	ccc	ddd	$\cdots$	baa	$\cdots$	caa	add	daa	bcc	cbb	bdd	$\cdots$	cdd	$\cdots$
70_{3013}	$\cdots$	$\cdots$	$\cdots$	$\cdots$	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
71_{3013}	aaa	$\cdots$	$\cdots$	$\cdots$	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
72_{3013}	aaa	bbb	$\cdots$	$\cdots$	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
73_{3013}	$\cdots$	$\cdots$	ccc	$\cdots$	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
74_{3013}	aaa	$\cdots$	ccc	$\cdots$	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
75_{3013}	aaa	bbb	ccc	$\cdots$	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
76_{3013}	aaa	bbb	$\cdots$	ddd	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
77_{3013}	aaa	bbb	ccc	ddd	abb	baa	$\cdots$	caa	add	daa	$\cdots$	cbb	bdd	dbb	cdd	$\cdots$
78_{3013}	$\cdots$	$\cdots$	$\cdots$	$\cdots$	abb	baa	acc	caa	add	daa	bcc	cbb	bdd	dbb	cdd	dcc
79_{3013}	aaa	$\cdots$	$\cdots$	$\cdots$	abb	baa	acc	caa	add	daa	bcc	cbb	bdd	dbb	cdd	dcc
80_{3013}	aaa	bbb	$\cdots$	$\cdots$	abb	baa	acc	caa	add	daa	bcc	cbb	bdd	dbb	cdd	dcc
81_{3013}	aaa	bbb	ccc	$\cdots$	abb	baa	acc	caa	add	daa	bcc	cbb	bdd	dbb	cdd	dcc
82_{3013}	aaa	bbb	ccc	ddd	abb	baa	acc	caa	add	daa	bcc	cbb	bdd	dbb	cdd	dcc

$$\mathrm{Cp}(\mathsf{1}_{3013}) = \begin{pmatrix}0\,0\,0\,0\\1\,1\,1\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{2}_{3013}) = \begin{pmatrix}1\,0\,0\,0\\0\,1\,1\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{3}_{3013}) = \begin{pmatrix}0\,1\,0\,0\\1\,0\,1\,1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{4}_{3013}) = \begin{pmatrix}1\,1\,0\,0\\0\,0\,1\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{5}_{3013}) = \begin{pmatrix}0\,0\,1\,0\\1\,1\,0\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{6}_{3013}) = \begin{pmatrix}1\,0\,1\,0\\0\,1\,0\,1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{7}_{3013}) = \begin{pmatrix}0\,1\,1\,0\\1\,0\,0\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{8}_{3013}) = \begin{pmatrix}1\,1\,1\,0\\0\,0\,0\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{9}_{3013}) = \begin{pmatrix}0\,0\,0\,1\\1\,1\,1\,0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{10}_{3013}) = \begin{pmatrix}1\,0\,0\,1\\0\,1\,1\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{11}_{3013}) = \begin{pmatrix}0\,1\,0\,1\\1\,0\,1\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{12}_{3013}) = \begin{pmatrix}1\,1\,0\,1\\0\,0\,1\,0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{13}_{3013}) = \begin{pmatrix}0\,0\,1\,1\\1\,1\,0\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{14}_{3013}) = \begin{pmatrix}1\,0\,1\,1\\0\,1\,0\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{15}_{3013}) = \begin{pmatrix}0\,1\,1\,1\\1\,0\,0\,0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{16}_{3013}) = \begin{pmatrix}1\,1\,1\,1\\0\,0\,0\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{17}_{3013}) = \begin{pmatrix}0\,0\,0\\2\,1\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{18}_{3013}) = \begin{pmatrix}1\,0\,0\\1\,1\,1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{19}_{3013}) = \begin{pmatrix}2\,0\,0\\0\,1\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{20}_{3013}) = \begin{pmatrix}0\,1\,0\\2\,0\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{21}_{3013}) = \begin{pmatrix}1\,1\,0\\1\,0\,1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{22}_{3013}) = \begin{pmatrix}2\,1\,0\\0\,0\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{23}_{3013}) = \begin{pmatrix}0\,0\,1\\2\,1\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{24}_{3013}) = \begin{pmatrix}1\,0\,1\\1\,1\,0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{25}_{3013}) = \begin{pmatrix}2\,0\,1\\0\,1\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{26}_{3013}) = \begin{pmatrix}0\,1\,1\\2\,0\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{27}_{3013}) = \begin{pmatrix}1\,1\,1\\1\,0\,0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{28}_{3013}) = \begin{pmatrix}2\,1\,1\\0\,0\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{29}_{3013}) = \begin{pmatrix}0\,0\,0\\1\,2\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{30}_{3013}) = \begin{pmatrix}0\,1\,0\\1\,1\,1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{31}_{3013}) = \begin{pmatrix}1\,0\,0\\0\,2\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{32}_{3013}) = \begin{pmatrix}1\,1\,0\\0\,1\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{33}_{3013}) = \begin{pmatrix}0\,2\,0\\1\,0\,1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{34}_{3013}) = \begin{pmatrix}1\,2\,0\\0\,0\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{35}_{3013}) = \begin{pmatrix}0\,0\,1\\1\,2\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{36}_{3013}) = \begin{pmatrix}0\,1\,1\\1\,1\,0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{37}_{3013}) = \begin{pmatrix}1\,0\,1\\0\,2\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{38}_{3013}) = \begin{pmatrix}1\,1\,1\\0\,1\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{39}_{3013}) = \begin{pmatrix}0\,2\,1\\1\,0\,0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{40}_{3013}) = \begin{pmatrix}1\,2\,1\\0\,0\,0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{41}_{3013}) = \begin{pmatrix}0\,0\,0\\1\,1\,2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{42}_{3013}) = \begin{pmatrix}0\,0\,1\\1\,1\,1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{43}_{3013}) = \begin{pmatrix}1\,0\,0\\0\,1\,2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{44}_{3013}) = \begin{pmatrix}1\,0\,1\\0\,1\,1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{45}_{3013}) = \begin{pmatrix}0\,1\,0\\1\,0\,2\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{46}_{3013}) = \begin{pmatrix}0&1&1\\1&0&1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{47}_{3013}) = \begin{pmatrix}1&1&0\\0&0&2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{48}_{3013}) = \begin{pmatrix}1&1&1\\0&0&1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{49}_{3013}) = \begin{pmatrix}0&0&2\\1&1&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{50}_{3013}) = \begin{pmatrix}1&0&2\\0&1&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{51}_{3013}) = \begin{pmatrix}0&1&2\\1&0&0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{52}_{3013}) = \begin{pmatrix}1&1&2\\0&0&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{53}_{3013}) = \begin{pmatrix}0&0\\3&1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{54}_{3013}) = \begin{pmatrix}1&0\\2&1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{55}_{3013}) = \begin{pmatrix}2&0\\1&1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{56}_{3013}) = \begin{pmatrix}3&0\\0&1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{57}_{3013}) = \begin{pmatrix}0&1\\3&0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{58}_{3013}) = \begin{pmatrix}1&1\\2&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{59}_{3013}) = \begin{pmatrix}2&1\\1&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{60}_{3013}) = \begin{pmatrix}3&1\\0&0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{61}_{3013}) = \begin{pmatrix}0&0\\2&2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{62}_{3013}) = \begin{pmatrix}0&1\\2&1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{63}_{3013}) = \begin{pmatrix}1&0\\1&2\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{64}_{3013}) = \begin{pmatrix}1&1\\1&1\end{pmatrix} \quad \mathrm{Cp}(\mathsf{65}_{3013}) = \begin{pmatrix}2&0\\0&2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{66}_{3013}) = \begin{pmatrix}2&1\\0&1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{67}_{3013}) = \begin{pmatrix}0&2\\2&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{68}_{3013}) = \begin{pmatrix}1&2\\1&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{69}_{3013}) = \begin{pmatrix}2&2\\0&0\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{70}_{3013}) = \begin{pmatrix}0&0\\1&3\end{pmatrix} \quad \mathrm{Cp}(\mathsf{71}_{3013}) = \begin{pmatrix}0&1\\1&2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{72}_{3013}) = \begin{pmatrix}0&2\\1&1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{73}_{3013}) = \begin{pmatrix}1&0\\0&3\end{pmatrix} \quad \mathrm{Cp}(\mathsf{74}_{3013}) = \begin{pmatrix}1&1\\0&2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{75}_{3013}) = \begin{pmatrix}1&2\\0&1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{76}_{3013}) = \begin{pmatrix}0&3\\1&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{77}_{3013}) = \begin{pmatrix}1&3\\0&0\end{pmatrix} \quad \mathrm{Cp}(\mathsf{78}_{3013}) = \begin{pmatrix}0\\4\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{79}_{3013}) = \begin{pmatrix}1\\3\end{pmatrix} \quad \mathrm{Cp}(\mathsf{80}_{3013}) = \begin{pmatrix}2\\2\end{pmatrix} \quad \mathrm{Cp}(\mathsf{81}_{3013}) = \begin{pmatrix}3\\1\end{pmatrix}$$

$$\mathrm{Cp}(\mathsf{82}_{3013}) = \begin{pmatrix}4\\0\end{pmatrix}$$

$$\mathsf{1} \cdots \mathsf{52}, \mathsf{61} \cdots \mathsf{69}_{3013} \in \mathsf{GRA} \cap \mathsf{fRRA}$$

$$\mathsf{53}, \mathsf{54}, \mathsf{55}, \mathsf{57}, \mathsf{58}, \mathsf{59}, \mathsf{70} \cdots \mathsf{75}, \mathsf{78} \cdots \mathsf{81}_{3013} \not\models (\mathrm{M})$$

$$\mathsf{53}, \mathsf{54}, \mathsf{55}, \mathsf{57}, \mathsf{58}, \mathsf{59}, \mathsf{70} \cdots \mathsf{75}, \mathsf{78} \cdots \mathsf{81}_{3013} \in \mathsf{RA} \sim (\mathsf{GRA} \cup \mathsf{RRA} \cup \mathsf{wRRA} \cup \mathsf{RA}_5)$$

$$\mathsf{56}, \mathsf{60}, \mathsf{76}, \mathsf{77}, \mathsf{82}_{3013} \in \mathsf{GRA} \sim \mathsf{fRRA}$$