Relational relevance algebras

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Semantics for relevance logic

Interpreting sentences as *sets* (unary relations) is—
Sound and complete semantics for propositional logic.
Interpretating sentences as binary relations is—
Sound for relevance logic
Incomplete for **R**
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R and **RM**

Pv is a countable set of **propositional variables**

The **connectives** are $\vee$, $\wedge$, $\circ$, $\to$, $\sim$

Sent = set of **sentences** = closure of Pv under the connectives

The **axioms of R** are (A1)–(A31)

The **axioms of RM** are (A1)–(A33)

R(**RM**) $\vdash A$ if A belongs to every set of sentences that contains the axioms of **R**(**RM**) and is closed under *modus ponens* and Adjunction.

Theorem.[Routley and Meyer (1973)] **R** $\vdash A$ iff A is derivable, using only *modus ponens* and adjunction, from those axioms among A1–A15 that explicitly contain connectives in A.

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Rules of deduction

Used in **R** and **RM**:

$A,\ A \to B \vdash\ B$	***modus ponens***
$A,\ B \vdash\ A \wedge B$	**Adjunction**

Other rules:

$A \to {\sim}B \vdash\ B \to {\sim}A$	**Contraposition**
${\sim}A,\ A \vee B \vdash\ B$	**Disjunctive Syll.**
$A \wedge B \to C,\ B \to C \vee A \vdash\ B \to C$	**Cut**
$A \to B \vdash (C \to A) \to (C \to B)$	**Prefixing**
$A \to B \vdash (B \to C) \to (A \to C)$	**Suffixing**
$A \to (B \to C) \vdash\ B \to ({\sim}C \to {\sim}A)$	**Cycling**

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$A \to A$	A1	(A1)
$A \wedge B \to A$	A5	(A2)
$A \wedge B \to B$	A6	(A3)
$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	A7	(A4)
$A \to A \vee B$	A8	(A5)
$B \to A \vee B$	A9	(A6)
$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	A10	(A7)
$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	A11	(A8)
$\sim\sim A \to A$	A13	(A9)

$$\sim(A \vee B) \to (\sim A \wedge \sim B) \tag{A10}$$

$$(\sim A \wedge \sim B) \to \sim(A \vee B) \tag{A11}$$

$$(A \to B) \to ((C \to A) \to (C \to B)) \tag{A12}$$

$$((A \to A) \to B) \to B \tag{A13}$$

$$A \to ((\sim B \to \sim A) \to B) \tag{A14}$$

$$A \to (\sim B \to \sim(A \to B)) \tag{A15}$$

$$(A \to (B \to C)) \to (\sim(A \to \sim B) \to C) \tag{A16}$$

$$A \circ B \to \sim(A \to \sim B) \tag{A17}$$

$$\sim(A \to \sim B) \to A \circ B \tag{A18}$$

$A \rightarrow (B \rightarrow (A \circ B))$		A14	(A19)
$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \circ B) \rightarrow C)$		A15	(A20)
$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$	c		(A21)
$(A \rightarrow {\sim}B) \rightarrow (B \rightarrow {\sim}A)$	c	A12	(A22)
$A \rightarrow ((A \rightarrow B) \rightarrow B)$	c	A2	(A23)
$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	c	A3	(A24)

$(A \to (A \to B)) \to (A \to B)$	d	A4	(A25)
$(A \to \sim A) \to \sim A$	d		(A26)
$(A \to (B \to C)) \to ((A \wedge B) \to C)$	d		(A27)
$(A \to B) \to (\sim A \vee B)$	d		(A28)
$(A \wedge (A \to B)) \to B$	d		(A29)
$((A \to B) \wedge (B \to C)) \to (A \to C)$	d		(A30)
$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	cd		(A31)
$A \to (A \to A)$	t		(A32)
$(A \to B) \to (A \to (A \to B))$	t		(A33)

Examples of theorems and non-theorems

$\mathbf{R} \nvdash A \to (B \to A)$

$\mathbf{R} \nvdash (A \wedge {\sim}A) \to B$

$\mathbf{R} \vdash B \vee {\sim}B$

$\mathbf{R} \nvdash B \to (A \to A)$

$\mathbf{R} \nvdash A \to (B \vee {\sim}B)$

$\mathbf{R} \vdash {\sim}(B \wedge {\sim}B)$

Definitions

U is a non-empty set
$U^2 = \{\langle x, y\rangle : x, y \in U\}$ = set of ordered pairs of elements of U
$A, B, C \subseteq U^2$

$$\begin{aligned}
A \cup B &= \{\langle x, y\rangle : \langle x, y\rangle \in A \text{ or } \langle x, y\rangle \in B\}\\
A \cap B &= \{\langle x, y\rangle : \langle x, y\rangle \in A \text{ and } \langle x, y\rangle \in B\}\\
A - B &= \{\langle x, y\rangle : \langle x, y\rangle \in A \text{ and } \langle x, y\rangle \notin B\}\\
A^{-1} &= \{\langle y, x\rangle : \langle x, y\rangle \in A\}\\
A|B &= \{\langle x, z\rangle : \exists_y(\langle x, y\rangle \in A \text{ and } \langle y, z\rangle \in B)\}\\
A \to B &= \{\langle x, y\rangle : x, y \in U, \forall_{z\in U}(\langle z, x\rangle \in A \implies \langle z, y\rangle \in B)\}
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More definitions

$\mathrm{Id} = \{\langle x, x\rangle : x \in U\}$	identity relation
$\mathrm{Di} = \{\langle x, y\rangle : x, y \in U \text{ and } x \neq y\}$	diversity relation
$\overline{A} = U^2 - A$	Boolean complement
$\sim A = U^2 - A^{-1}$	De Morgan complement
$A \circ B = B \mid A$	composition/fusion
$A \dagger B = \sim(\sim B \mid \sim A)$	relative sum
$A \rightarrow B = \sim(\sim B \mid A)$	residual

A is **symmetric** if $A = A^{-1}$, **transitive** if $A \circ A \subseteq A$, **dense** if $A \subseteq A \circ A$

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Computational lemmas

$$\mathrm{Id} \to A = A$$

$$A \subseteq B \quad \text{iff} \quad \mathrm{Id} \subseteq A \to B$$

$$A \to (B \to C) = B|A \to C = A \circ B \to C$$

$$A|(A \to B) \subseteq B$$

$$(A \to B) \circ A \subseteq B$$

$$(A \to B)|{\sim}B \subseteq {\sim}A$$

$${\sim}B \circ (A \to B) \subseteq {\sim}A$$

$$A \subseteq B \text{ implies } B \to C \subseteq A \to C$$

$$A \subseteq B \text{ implies } C \to A \subseteq C \to B$$

$\mathfrak{R} = \langle R, \cup, \cap, \circ, \rightarrow, \sim \rangle$ is a **relational relevance algebra** if

- R is non-empty set of binary relations on $U \neq \emptyset$
- R is closed under $\cup$, $\cap$, $\circ$, $\rightarrow$, $\sim$

$\mathfrak{R}$ is **commutative** if $A \circ B = B \circ A$ for all $A, B \in R$.

$\mathfrak{R}$ is **dense**, **transitive**, **symmetric** if every relation in $\mathfrak{R}$ is dense, transitive, symmetric, resp.

R=class of relational relevance algebras.

R^{cd}=class of commutative dense relational relevance algebras.

R^{cdt}=class of commutative dense transitive relational relevance algebras.

Define R^{c}, R^{d}, R^{dt}, R^{ct}, R^{t} similarly.

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Assume: $\mathfrak{R} \in \mathrm{R}$ is a relational relevance algebra on U, Id is the identity relation on U, $A \in \underline{\mathrm{Sent}}$, $K \subseteq \mathrm{R}$

$$\begin{aligned}
\mathfrak{R} \models A \quad & \text{iff} \quad A \text{ is } \textbf{valid in } \mathfrak{R} \\
& \text{iff} \quad \mathrm{Id} \subseteq H(A) \text{ for every homomorphism} \\
& \qquad\quad H : \langle \underline{\mathrm{Sent}}, \vee, \wedge, \circ, \rightarrow, \sim \rangle \rightarrow \mathfrak{R} \\
K \models A \quad & \text{iff} \quad A \text{ is } \textbf{valid in } K \\
& \text{iff} \quad A \text{ is valid in every algebra in } K
\end{aligned}$$

K-**logic** is the set of sentences valid in all algebras in K.

Assume: $\mathfrak{R} \in \mathsf{R}$ is a relational relevance algebra on U, Id is the identity relation on U, $A \in \underline{\mathrm{Sent}}$, $K \subseteq \mathsf{R}$

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Sound and complete?

Soundness theorems:

1. if **R** $\vdash A$ then $\mathsf{R}^{cd} \models A$
2. if **RM** $\vdash A$ then $\mathsf{R}^{cdt} \models A$

Completeness?

Q1 if **R** $\nvdash A$, is A not valid in some $\mathfrak{R} \in \mathsf{R}^{cd}$?

Q2 if **RM** $\nvdash A$, is A not valid in some $\mathfrak{R} \in \mathsf{R}^{cdt}$?

(Q1): “no” because there is no finite axiomatization of the sentences that are valid in R^{cd} (Mikulás 2008).

(Q2): “yes” because the finite normal Sugihara matrices are isomorphic to algebras in R^{cdt}.

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Completeness?

Q1 if **R** $\nvdash A$, is A not valid in some $\mathfrak{R} \in \mathrm{R}^{cd}$?

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(Q1): “no” because there is no finite axiomatization of the sentences that are valid in R^{cd} (Mikulás 2008).
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Sound and complete?

Soundness theorems:

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Soundness results

—Axioms are valid in relational relevance algebras:

$$\begin{aligned} \mathsf{R} &\models \text{(A1)–(A20)} \\ \mathsf{R}^{c} &\models \text{(A21)–(A24)} \\ \mathsf{R}^{d} &\models \text{(A25)–(A30)} \\ \mathsf{R}^{cd} &\models \text{(A31)} \\ \mathsf{R}^{t} &\models \text{(A32), (A33)} \end{aligned}$$

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$$\mathsf{R}^{cd} \models \text{(A1)–(A31)}$$

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Rules of deduction preserve validity in R because

if $\mathrm{Id} \subseteq A$ and $\mathrm{Id} \subseteq A \to B$ then $\mathrm{Id} \subseteq B$

if $\mathrm{Id} \subseteq A$ and $\mathrm{Id} \subseteq B$ then $\mathrm{Id} \subseteq A \cap B$

if $\mathrm{Id} \subseteq A \to {\sim}B$ then $\mathrm{Id} \subseteq B \to {\sim}A$

if $\mathrm{Id} \subseteq {\sim}A$ and $\mathrm{Id} \subseteq A \cup B$ then $\mathrm{Id} \subseteq B$

if $\mathrm{Id} \subseteq A \cap B \to C$ and $\mathrm{Id} \subseteq B \to C \cup A$ then $\mathrm{Id} \subseteq B \to C$

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(A1)–(A11) are valid in R because

$\mathrm{Id} \subseteq A \to A$

$\mathrm{Id} \subseteq A \cap B \to A$ $\qquad$ $\mathrm{Id} \subseteq A \cap B \to B$

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$\mathrm{Id} \subseteq {\sim}{\sim}A \to A$

$\mathrm{Id} \subseteq {\sim}(A \cup B) \to ({\sim}A \cap {\sim}B)$

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(A12)–(A20) are valid in R because

$$\mathrm{Id} \subseteq (A \to B) \to ((C \to A) \to (C \to B))$$

$$\mathrm{Id} \subseteq ((A \to A) \to B) \to B$$

$$\mathrm{Id} \subseteq A \to ((\sim B \to \sim A) \to B)$$

$$\mathrm{Id} \subseteq A \to (\sim B \to \sim(A \to B))$$

$$\mathrm{Id} \subseteq (A \to (B \to C)) \to (\sim(A \to \sim B) \to C)$$

$$\mathrm{Id} \subseteq A \circ B \to \sim(A \to \sim B)$$

$$\mathrm{Id} \subseteq \sim(A \to \sim B) \to A \circ B$$

$$\mathrm{Id} \subseteq A \to (B \to (A \circ B))$$

$$\mathrm{Id} \subseteq (A \to (B \to C)) \to ((A \circ B) \to C)$$

(A21)–(A24) are valid in R^c because

if $\{A, B\}$ is commutative then

$$\mathrm{Id} \subseteq (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$$

if $\{A, B\}$ is commutative then $\mathrm{Id} \subseteq (A \rightarrow {\sim}B) \rightarrow (B \rightarrow {\sim}A)$

if $\{A, A \rightarrow B\}$ is commutative then $\mathrm{Id} \subseteq A \rightarrow ((A \rightarrow B) \rightarrow B)$

if $\{B \rightarrow C, A \rightarrow B\}$ is commutative then

$$\mathrm{Id} \subseteq (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

(A25)–(A30) are valid in R^d because

if A is dense then $\mathrm{Id} \subseteq (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

if A is dense then $\mathrm{Id} \subseteq (A \rightarrow {\sim}A) \rightarrow {\sim}A$

if $A \cap B$ is dense then $\mathrm{Id} \subseteq (A \rightarrow (B \rightarrow C)) \rightarrow ((A \cap B) \rightarrow C)$

if $A \cap {\sim}B$ is dense then $\mathrm{Id} \subseteq (A \rightarrow B) \rightarrow ({\sim}A \cup B)$

if $A \cap (A \rightarrow B)$ is dense then $\mathrm{Id} \subseteq (A \cap (A \rightarrow B)) \rightarrow B$

if $(A \rightarrow B) \cap (B \rightarrow C)$ is dense then

$$\mathrm{Id} \subseteq ((A \rightarrow B) \cap (B \rightarrow C)) \rightarrow (A \rightarrow C)$$

(A31) is valid in R^{cd} because

if $\{A, A \rightarrow B\}$ is commutative and A is dense then

$$\mathrm{Id} \subseteq (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

(A32) and (A33) are valid in R^{t} because

if A is transitive then $\mathrm{Id} \subseteq A \rightarrow (A \rightarrow A)$

if A is transitive then $\mathrm{Id} \subseteq (A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))$

Thesis

$\boxed{\text{Relevance logic is } \mathsf{R}^{cd}\text{-logic}}$

Theorem. (Thesis) **R** is a partial (incomplete) axiomatization of relevance logic.

Theorem. **RM** is R^{cdt}-logic.

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Theorem. **RM** is R^{cdt}-logic.

Incompleteness of **R**

If U is a non-empty set and $A, B, C, D, E, F, G \subseteq U^2$ then

$$\mathrm{Id} \subseteq A|B \cap C|D \cap E|F \to \qquad \text{(L)}$$

$$A|\left(A^{-1}|C \cap B|D^{-1} \cap (A^{-1}|E \cap B|F^{-1})|(E^{-1}|C \cap F|D^{-1})\right)|D$$

$$\mathrm{Id} \subseteq A|B \cap C|D \cap E|F \to \qquad \text{(L}'\text{)}$$

$$\left((A \cap {\sim}A)|B \cap C|D \cap E|F\right) \cup \left(A|B \cap C|(D \cap {\sim}D) \cap E|F\right)$$

$$\cup \left(A|B \cap C|D \cap (E \cap {\sim}E)|F\right) \cup \left(A|B \cap C|D \cap E|(F \cap {\sim}F)\right)$$

$$\cup A|\left(A|C \cap B|D \cap (A|E \cap B|F)|(E|C \cap F|D)\right)|D$$

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$$\begin{aligned} &\mathrm{Id} \subseteq A|B \cap C|D \cap E|F \to \\ &\quad \Big((A \cap {\sim}A)|B \cap C|D \cap E|F\Big) \cup \Big(A|B \cap C|(D \cap {\sim}D) \cap E|F\Big) \\ &\quad \cup \Big(A|B \cap C|D \cap (E \cap {\sim}E)|F\Big) \cup \Big(A|B \cap C|D \cap E|(F \cap {\sim}F)\Big) \\ &\quad \cup A|\Big(A|C \cap B|D \cap (A|E \cap B|F)|(E|C \cap F|D)\Big)|D \end{aligned} \tag{L$'$}$$

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If U is a non-empty set and $A, B, C, D, E, F, G \subseteq U^2$ then

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Incompleteness of **R**

$$A|B = B \circ A = {\sim}(B \to {\sim}A)$$

$$\begin{aligned} & A|B \wedge C|D \wedge E|F \to \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{L}'') \\ & \Big((A \wedge {\sim}A)|B \wedge C|D \wedge E|F\Big) \vee \Big(A|B \wedge C|(D \wedge {\sim}D) \wedge E|F\Big) \\ \vee\ & \Big(A|B \wedge C|D \wedge (E \wedge {\sim}E)|F\Big) \vee \Big(A|B \wedge C|D \wedge E|(F \wedge {\sim}F)\Big) \\ \vee\ & A|\Big(A|C \wedge B|D \wedge (A|E \wedge B|F)|(E|C \wedge F|D)\Big)|D \end{aligned}$$

(L″) is valid in every relational relevance algebra (in R).

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Routley-Meyer semantics

$\mathfrak{K} = \langle K, R, ^*, 0\rangle$ is a **relevant model structure** if $\emptyset \neq K$, $R \subseteq K^3$, $^* : K \to K$, $0 \in K$, and (p1)–(p6) hold for $a, b, c \in K$:

$$\begin{array}{lll}
R0aa & \text{0-reflexivity} & \text{(p1)} \\
Raaa & \text{density} & \text{(p2)} \\
R^2abcd \implies R^2acdb & \text{Pasch} & \text{(p3)} \\
R^20abc \implies Rabc & \text{0-cancellation} & \text{(p4)} \\
Rabc \implies Rac^*b^* & & \text{(p5)} \\
a^{**} = a & \text{involution} & \text{(p6)} \\
R^2abcd \quad \text{iff} \quad \exists_x(Rabx, Rxcd, x \in K) & & \text{(d1)}
\end{array}$$

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Routley-Meyer Completeness

$X \subseteq K$ is **0-closed** if $y \in X$ whenever $x \in X$ and $R0xy$.

A **valuation** on $\mathfrak{K}$ is a function $\nu : \underline{\text{Sent}} \to Sb(K)$ such that

$$
\begin{aligned}
\nu(A) &\text{ is 0-closed if } A \in \text{Pv}, \\
\nu(A \wedge B) &= \nu(A) \cap \nu(B) \\
\nu(A \vee B) &= \nu(A) \cup \nu(B) \\
\nu(A \circ B) &= \{c : (\exists a, b \in K)(Rabc \text{ and } a \in \nu(A) \text{ and } b \in \nu(B))\} \\
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A is **valid in** $\mathfrak{K}$ if $0 \in \nu(A)$ for every valuation ν on $\mathfrak{K}$

Theorem. (Routley-Meyer) $\mathbf{R} \nvdash A$ iff A is valid in every relevant model structure.

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Incompleteness of **R**

Theorem. R $\nvdash$ (L″). Proof. Let $\mathfrak{K}_{28}$ be the relevant model structure defined as follows:

$$\begin{aligned}
\mathfrak{K}_{28} &= \langle K, R_{28}, {}^*, 0\rangle \\
K &= \{0, a, b, c\} \\
x^* &= x \text{ for every } x \in K \\
R_{28} &= [0,0,0] \cup [a,a,a] \cup [b,b,b] \cup [c,c,c] \cup \\
&\qquad [0,a,a] \cup [0,b,b] \cup [0,c,c] \cup \\
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&\qquad [a,b,b] \cup [c,a,a] \cup [b,c,c] \cup [a,b,c] \\
[a,b,c] &= \{abc, c^*ab^*, bc^*a^*, a^*cb, cb^*a, b^*a^*c^*\}
\end{aligned}$$

Incompleteness of **R**

$\mathfrak{K}_{28} \not\models (L'')$ because $0 \notin \nu(L'')$ if ν is a valuation such that

X	A	B	C	D	E	F	G
$\nu(X)$	$\{a\}$	$\{a\}$	$\{c\}$	$\{b\}$	$\{a\}$	$\{c\}$	$\{a\}$

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Sugihara matrices

Sugihara matrices $\mathbf{S}_n = \langle S_n, \vee, \wedge, \circ, \rightarrow, \sim\rangle$ for $1 \leq n < \omega$
—introduced by Sugihara (1955)
—simplified by Anderson and Belnap (1975, §26.9)
—used in completeness theorems for **RM** by R. K. Meyer
$S_n = \{-k, \dots, -1, 1, \dots, k\}$ if $n = 2k$, $k > 0$
$S_n = \{-k, \dots, -1, 0, 1, \dots, k\}$ if $n = 2k + 1$, $k \geq 0$
$S_1 = \{0\}$, $S_2 = \{-1, 1\}$, $S_3 = \{-1, 0, 1\}$, $S_4 = \{-2, -1, 1, 2\}$
$i \wedge j$ is the minimum of i and j
$i \vee j$ is the maximum of i and j
$\sim$ is multiplication by -1: $\sim 0 = 0$, $\sim i = -i$, $\sim(-i) = i$

Sugihara multiplication/fusion

$$\begin{aligned} -i \circ -j &= -\max(i,j) \\ -i \circ j &= \begin{cases} -i & \text{if } j \leq i \\ j & \text{if } i < j \end{cases} \\ i \circ j &= \max(i,j) \\ i \to j &= \sim(i \circ \sim j) \end{aligned}$$

$i \circ j$ is the maximum of i and j under the linear ordering of S_n that begins: $0 < 1 < -1 < 2 < -2 < 3 < -3 < 4 < -4 < \cdots$.

S_n is normal iff $0 \notin S_n$ iff n is even

Table for $\circ$ in $\mathbf{S}_8$

$\circ$	-4	-3	-2	-1	1	2	3	4
-4	-4	-4	-4	-4	-4	-4	-4	-4
-3	-4	-3	-3	-3	-3	-3	-3	4
-2	-4	-3	-2	-2	-2	-2	3	4
-1	-4	-3	-2	-1	-1	2	3	4
1	-4	-3	-2	-1	1	2	3	4
2	-4	-3	-2	2	2	2	3	4
3	-4	-3	3	3	3	3	3	4
4	-4	4	4	4	4	4	4	4

Table for $\circ$ in $\mathbf{S}_9$

$\circ$	-4	-3	-2	-1	0	1	2	3	4
-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
-3	-4	-3	-3	-3	-3	-3	-3	-3	4
-2	-4	-3	-2	-2	-2	-2	-2	3	4
-1	-4	-3	-2	-1	-1	-1	2	3	4
0	-4	-3	-2	-1	0	1	2	3	4
1	-4	-3	-2	-1	1	1	2	3	4
2	-4	-3	-2	2	2	2	2	3	4
3	-4	-3	3	3	3	3	3	3	4
4	-4	4	4	4	4	4	4	4	4

Sugihara completeness for **RM**

A is **valid in $\mathbf{S}_n$** if $H(A) \in \{1, \ldots, n\}$ for every homomorphism

$$H : \langle \underline{\mathsf{Sent}}, \vee, \wedge, \circ, \rightarrow, \sim \rangle \rightarrow \mathbf{S}_n$$

Theorem (Meyer; see (Anderson and Belnap 1975, Cor. 3.1)) If *A* has no more than *n* propositional variables then

$$\mathbf{RM} \vdash A \quad \text{iff} \quad A \text{ is valid in } \mathbf{S}_n$$

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Sugihara matrices are in R^{cdt}

Theorem. $\mathbf{S}_{2n+2} \cong \mathbf{T}_n \in \mathsf{R}^{cdt}$

$$\mathbb{Q} = \text{ set of rational numbers}$$

$$\mathbb{Q}^n = \{\langle q_1, \dots, q_n\rangle : q_1, \dots, q_n \in \mathbb{Q}\}$$

$$\mathrm{Id} = \{\langle q, q\rangle : q \in \mathbb{Q}^n\}$$

$$\langle q, q'\rangle \in L_1 \quad \text{iff} \quad q_1 < q'_1$$

$$\langle q, q'\rangle \in L_i \quad \text{iff} \quad \langle q_1, \dots, q_{i-1}\rangle = \langle q'_1, \dots, q'_{i-1}\rangle \text{ and } q_i < q'_i$$

$$\mathcal{L}_n = \{\mathrm{Id}, L_1, L_1^{-1}, \dots, L_n, L_n^{-1}\}, \text{ a partition of } \mathbb{Q}^n \times \mathbb{Q}^n,$$

$$\text{atoms of } A_n = \left\{\bigcup S : S \subseteq \mathcal{L}_n\right\}, \text{ a finite Boolean algebra}$$

A_n is closed under conversion $^{-1}$

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A_n is closed under relative multiplication:

$$\begin{aligned}
\mathrm{Id} &= \mathrm{Id}|\mathrm{Id} \\
L_i &= L_i|\mathrm{Id} = \mathrm{Id}|L_i \\
L_i^{-1} &= L_i^{-1}|\mathrm{Id} = \mathrm{Id}|L_i^{-1} \\
L_i|L_j &= L_{\min(i,j)} \\
L_i^{-1}|L_j^{-1} &= L_{\min(i,j)}^{-1} \\
L_j^{-1}|L_i = L_i|L_j^{-1} &= \begin{cases} L_i & \text{if } i < j \\ L_j^{-1} & \text{if } j < i \\ \mathrm{Id} \cup \bigcup_{i=j \le k \le n} (L_k \cup L_k^{-1}) & \text{if } i = j \end{cases}
\end{aligned}$$

If $J \subseteq \{1, 2, \ldots, n\}$ and $1 \leq i \leq n$ then

$$\begin{aligned} L_\emptyset &= \emptyset & L_\emptyset^{-1} &= \emptyset \\ L_J &= \bigcup_{i \in J} L_i & L_J^{-1} &= \bigcup_{i \in J} L_i^{-1} \quad \text{if } \emptyset \neq J \\ [1, i] &= \{1, 2, \ldots, i-1, i\} & [i, n] &= \{i, i+1, \ldots, n-1, n\} \end{aligned}$$

Define $T : S_{2n+2} \to Sb(\mathbb{Q}^n \times \mathbb{Q}^n)$ by

$$\begin{aligned} T_{-n-1} &= \emptyset \\ T_{-i} &= L_{[1,n+1-i]} \quad \text{if } 1 \leq i \leq n \\ T_1 &= L_{[1,n]} \cup \mathrm{Id} \\ T_i &= L_{[1,n]} \cup \mathrm{Id} \cup L^{-1}_{[n+2-i,n]} \quad \text{if } 2 \leq i \leq n+1 \end{aligned}$$

Example

$$\emptyset = T_{-n-1} \subseteq T_{-n} \subseteq \cdots \subseteq T_{-1} \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq T_{n+1} = \mathbb{Q}^n \times \mathbb{Q}^n$$

$$\begin{aligned} T_{-4} &= \emptyset & T_4 &= L_{\{1,2,3\}} \cup \mathrm{Id} \cup L^{-1}_{\{3,2,1\}} \\ T_{-3} &= L_{\{1\}} & T_3 &= L_{\{1,2,3\}} \cup \mathrm{Id} \cup L^{-1}_{\{3,2\}} \\ T_{-2} &= L_{\{1,2\}} & T_2 &= L_{\{1,2,3\}} \cup \mathrm{Id} \cup L^{-1}_{\{3\}} \\ T_{-1} &= L_{\{1,2,3\}} & T_1 &= L_{\{1,2,3\}} \cup \mathrm{Id} \end{aligned}$$

closure

Let $\mathcal{T}_n = \{T_{-n-1}, \cdots, T_{-1}, T_1, \cdots, T_{n+1}\}$.

$\mathcal{T}_n$ is closed under union and intersection

$\mathcal{T}_n$ is closed under converse-complementation because $\sim(T_i) = T_{-i} = T_{\sim(i)}$ for all $i \in S_{2n+2}$

$\mathcal{T}_n$ is closed under relative multiplication because

$$T_{-i}|T_{-j} = T_{-\max(i,j)}$$

$$T_{-i}|T_j = \begin{cases} T_{-i} & \text{if } i \geq j \\ T_j & \text{if } i < j \end{cases}$$

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$\mathbf{T}_n = \langle \mathcal{T}_n, \cup, \cap, \circ, \to, \sim \rangle$ is a finite commutative dense transitive relational relevance algebra
$\mathbf{S}_{2n+2} \cong \mathbf{T}_n \in \mathsf{R}^{cdt}$
If A has no more than $2n+2$ propositional variables, then

$$\mathbf{RM} \vdash A \quad \text{iff} \quad A \text{ is valid in } \mathbf{T}_n$$

$\mathbf{RM} \vdash A$ iff $\mathsf{R}^{cdt} \models A$
RM is R^{cdt}-logic

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