Some problems about coloring the edges of a complete graph.

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Start with a set $\mathcal{T}$ of edge-colored triangles. For example, $\mathcal{T}$ could be

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        /\                          /\
       /  \                        /  \
      /    \                      /    \
     /      \red                 /      \red
    /red     \                  /red     \
   /          \                /          \
  /            \              /            \
  ------red-----              -----green--->
```

Every color is directed or not (do not use both a directed red and an undirected red). For example, —green—> is directed and —red— is undirected. Think of the triangles in $\mathcal{T}$ as tiles. There is an unlimited supply of each type of tile.

Problem. Given a (possibly infinite) number n, can K_n (the complete graph on n points) be "tiled" with edge-colored triangles from $\mathcal{T}$ so that (1) for every three points in K_n, the tile they determine has colors that match one of the tiles in $\mathcal{T}$, and (2) whatever could occur, must occur. Details: (2a) From every point there emerges an edge of every possible color. For example, if P is in K_n then there are points Q,R,S such that P —red— Q, P —green—> R, and P <——green— S. (2b) If the color of an edge PQ in K_n matches an edge of a tile Δ in $\mathcal{T}$, then there is a point R in K_n such that the colors of PQR match Δ. For example, if P —red— Q occurs in K_n, then (because of the red-red-red tile in $\mathcal{T}$) there is an R such that P —red— R —red— Q, and (because of the red-red-green tile in $\mathcal{T}$) there is some S in K_n such that P —green—> S —red— Q, some T in K_n such that P —red— T <—green— Q, some U in K_n such that P < —green— U —red— Q, and some V in K_n such that P —red— V —green—> Q, corresponding to the four ways that a red edge can be matched with a red edge in the red-red-green tile.

Theorem 1. One directed color: green. There are only two possible green-green-green tiles, called the "3-cycle" and the "3-chain".

(1) K_n is tilable by the 3-cycle iff $n = 3$.

(2) K_n is tilable by the 3-chain iff n is infinite.

(3) K_n is tilable by the 3-cycle and 3-chain iff $n = 7$ or $9 \leq n$.

Theorem 2. Two undirected colors: red and blue.

(1) K_n is tilable by red-red-red iff $n \geq 3$.

(2) No K_n is tilable by red-red-red and blue-blue-blue.

(3) K_n is tilable by red-red-blue iff $n = 4$.

(4) K_n is tilable by red-red-blue and blue-blue-red iff $n = 5$.

Theorem 3. Three undirected colors: red, blue, and yellow.

(1) K_n is tilable by red-red-blue, blue-blue-red, red-red-yellow, yellow-yellow-red, blue-blue-yellow, yellow-yellow-blue, and red-blue-yellow iff $n \in \{13, 16\}$.

(2) K_n is tilable by red-red-blue, blue-blue-red, red-red-yellow, yellow-yellow-red, blue-blue-yellow, yellow-yellow-blue, red-red-red, blue-blue-blue, yellow-yellow-yellow (all tiles with exactly one or exactly two colors) iff n is infinite.

Open Problem. Given k undirected colors, $k \geq 3$, and $\mathcal{T}$ consisting of all tiles involving exactly two colors or exactly three colors, is there an n such that K_n is tilable? K_n is not tilable if n is larger than a number $r(k)$ that exists by Ramsey's Theorem. YES, if $k = 3$ by Theorem **3**(1). YES, if $k = 4, 5$ by S. Comer, using finite fields, in "Color schemes forbidding monochromatic triangles", *Congressus Numerantium* 1983, pp. 231–236. YES, for all sufficiently large n, by Trotter, Szemeredi, and Erdös, except their proof was wrong, so the problem is still open for $k \geq 6$.

Theorem 4. Any finite number of directed and undirected colors. If every possible triangle involving a particular color (called the "flexible color") appears in T then K_n is tilable for every *infinite* n.

Open Problem. Can the theorem be improved to conclude with "then K_n is tilable for some $n < \omega$"? (The Flexible Atom Conjecture: YES)