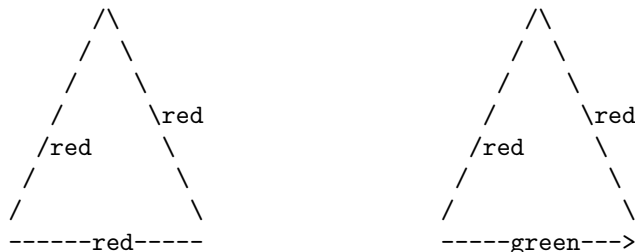


Some problems about coloring the edges of a complete graph.

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Start with a set \mathcal{T} of edge-colored triangles. For example, \mathcal{T} could be



Every color is directed or not (do not use both a directed red and an undirected red). For example, ---green--> is directed and ---red--- is undirected. Think of the triangles in \mathcal{T} as tiles. There is an unlimited supply of each type of tile.

Problem. Given a (possibly infinite) number n , can K_n (the complete graph on n points) be “tiled” with edge-colored triangles from \mathcal{T} so that (1) for every three points in K_n , the tile they determine has colors that match one of the tiles in \mathcal{T} , and (2) whatever could occur, must occur. Details: (2a) From every point there emerges an edge of every possible color. For example, if P is in K_n then there are points Q, R, S such that $P \text{---red---} Q$, $P \text{---green-->} R$, and $P \text{<---green---} S$. (2b) If the color of an edge PQ in K_n matches an edge of a tile Δ in \mathcal{T} , then there is a point R in K_n such that the colors of PQR match Δ . For example, if $P \text{---red---} Q$ occurs in K_n , then (because of the red-red-red tile in \mathcal{T}) there is an R such that $P \text{---red---} R \text{---red---} Q$, and (because of the red-red-green tile in \mathcal{T}) there is some S in K_n such that $P \text{---green-->} S \text{---red---} Q$, some T in K_n such that $P \text{---red---} T \text{<---green---} Q$, some U in K_n such that $P \text{<---green---} U \text{---red---} Q$, and some V in K_n such that $P \text{---red---} V \text{---green-->} Q$, corresponding to the four ways that a red edge can be matched with a red edge in the red-red-green tile.

Theorem 1. One directed color: green. There are only two possible green-green-green tiles, called the “3-cycle” and the “3-chain”.

- (1) K_n is tilable by the 3-cycle iff $n = 3$.
- (2) K_n is tilable by the 3-chain iff n is infinite.
- (3) K_n is tilable by the 3-cycle and 3-chain iff $n = 7$ or $9 \leq n$.

Theorem 2. Two undirected colors: red and blue.

- (1) K_n is tilable by red-red-red iff $n \geq 3$.
- (2) No K_n is tilable by red-red-red and blue-blue-blue.
- (3) K_n is tilable by red-red-blue iff $n = 4$.
- (4) K_n is tilable by red-red-blue and blue-blue-red iff $n = 5$.

Theorem 3. Three undirected colors: red, blue, and yellow.

- (1) K_n is tilable by red-red-blue, blue-blue-red, red-red-yellow, yellow-yellow-red, blue-blue-yellow, yellow-yellow-blue, and red-blue-yellow iff $n \in \{13, 16\}$.
- (2) K_n is tilable by red-red-blue, blue-blue-red, red-red-yellow, yellow-yellow-red, blue-blue-yellow, yellow-yellow-blue, red-red-red, blue-blue-blue, yellow-yellow-yellow (all tiles with exactly one or exactly two colors) iff n is infinite.

Open Problem. Given k undirected colors, $k \geq 3$, and \mathcal{T} consisting of all tiles involving exactly two colors or exactly three colors, is there an n such that K_n is tilable? K_n is not tilable if n is larger than a number $r(k)$ that exists by Ramsey’s Theorem. YES, if $k = 3$ by Theorem 3(1). YES, if $k = 4, 5$ by S. Comer, using finite fields, in “Color schemes forbidding monochromatic triangles”, *Congressus Numerantium* 1983, pp. 231–236. YES, for all sufficiently large n , by Trotter, Szemerédi, and Erdős, except their proof was wrong, so the problem is still open for $k \geq 6$.

Theorem 4. Any finite number of directed and undirected colors. If every possible triangle involving a particular color (called the “flexible color”) appears in \mathcal{T} then K_n is tilable for every *infinite* n .

Open Problem. Can the theorem be improved to conclude with “then K_n is tilable for some $n < \omega$ ”? (The Flexible Atom Conjecture: YES)