NONREPRESENTABLE RELATION ALGEBRAS GENERATED BY FUNCTIONAL ELEMENTS

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Abstract. There are nonrepresentable relation algebras generated by their functional elements. This solves a problem posed many years ago. The number of generating functional elements can be as low as 2. This leaves open the problem whether there is a nonrepresentable relation algebra generated by a single functional element.

1. Introduction

An element p in a relation algebra is said to be **functional** if $\breve{p};p \leq 1'$. In case p is a set of ordered pairs (in a proper relation algebra), this equation asserts that p is a function, in the sense that any two pairs in p having the same first element must also have the same second element. Several representation results for relation algebras involve functional elements. For example, Jónsson and Tarski [2] proved that a relation algebra is representable if its Boolean unit element is the join of a finite number of functional elements. Tarski and I [3] proved that the finiteness assumption can be dropped: if the join of (any number of) functional elements of a relation algebra is the Boolean unit, then the relation algebra must be representable. Tarski [9] proved that a relation algebra is representable whenever its Boolean unit has the form $\breve{p};q$ for some functional elements p and q. I showed that representability still holds if the Boolean unit is merely the join of such elements [5], [4]. Here are two problems about relation algebras generated by functional elements from a list dated June 17, 1975.

- (Tarski) What is the structure of the relation algebra generated by two biunique functional elements? Is it atomic?
- ([7, Prob. **P8**]) If a relation algebra is generated by its functional elements, must it be representable?

For the first problem, one should know that p is a biunique functional element if both $\breve{p};p \leq 1'$ and $p;\breve{p} \leq 1'$. The first problem was suggested by Tarski in a conversation around 1974. Its wording suggests that Tarski knew the structure of the relation algebra generated by a single biunique functional element, and that he knew it was atomic, although I don't remember him saying so. I asked myself the second question around the same time. It seemed like a natural question with possibly a positive answer, given that so many representation theorems involve functional elements. However, I show in this paper that the answer to the second problem is "no".

2. The complex algebra of a ternary relation.

Suppose that T is a ternary relation. Let U be the field of T, that is,

$$U := \{x : \exists_y \, \exists_z \, (Txyz \text{ or } Tyxz \text{ or } Tyzx)\}. \tag{1}$$

Let $Sb\,(U)$ is the set of all subsets of U. Let $\mathfrak{Bl}\,(U) = \langle Sb\,(U)\,, \cup, ^{-}\rangle$ be the complete atomic Boolean algebra of all subsets of U, where $\overline{X} = \{u : u \in U, u \notin X\}$ for all $X \subseteq U$. For any subsets

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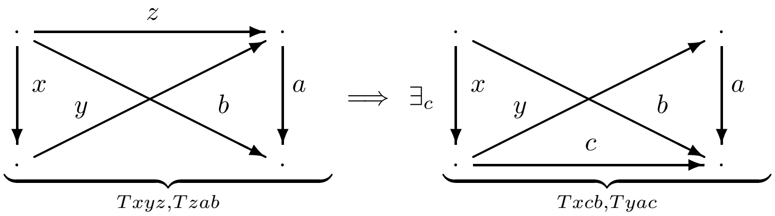


FIGURE 1. The associativity condition (A).

$X, Y \subseteq U$, define the product $X;Y$ by

$$X;Y = \{c : \exists_x \exists_y (x \in X, y \in Y, Txyc)\}.$$

Define the binary relation $S \subseteq U^2$ by

$$S := \{\langle a, b\rangle : a, b \in U, \forall_x \forall_y ((Taxy \iff Tbyx), (Txay \iff Tybx))\}. \tag{2}$$

By the form of its definition, S must be a symmetric relation. Use S to define $\breve{X}$ for every $X \subseteq U$ by

$$\breve{X} = \{b : \exists_x (Sxb, x \in X)\}$$

Define a subset $I \subseteq U$ by

$$I := \{a : a \in U, \forall_x \forall_y ((Taxy \text{ or } Txay) \implies x = y)\}. \tag{3}$$

The **complex algebra** of T is the algebra

$$\mathfrak{Cm}T = \langle Sb(U), \cup, {}^-, ;, {}^\smile, I\rangle .$$

The **Boolean part** of $\mathfrak{Cm}T$ is $\mathfrak{Bl}(U)$. The complex algebra $\mathfrak{Cm}T$ is a relation algebra when certain elementary conditions are satisfied by T.

Theorem 1. [6, Th. 2.2] *Suppose T is a ternary relation. Let U, S, and I satisfy* (1), (2), (3). *Consider the following statements about T:*

(C) $\forall_a (a \in U \implies \exists_b Sab)$,
(I) $\forall_a (a \in U \implies \exists_i (i \in I, Tiaa))$,
(A) $\forall_x \forall_y \forall_z \forall_a \forall_b (Txyz, Tzab \implies \exists_c (Txcb, Tyac))$,

Then

- *the Boolean part of $\mathfrak{Cm}T$ is a complete atomic Boolean algebra,*
- *the operators ; and $^\smile$ are normal and completely additive:*

$$\begin{aligned} &\emptyset;X = X;\emptyset = \emptyset \text{ for all } X \subseteq U,\\ &\breve{\emptyset} = \emptyset,\\ &\bigcup\mathcal{X};\bigcup\mathcal{Y} = \bigcup\{X;Y : X \in \mathcal{X}, Y \in \mathcal{Y}\} \text{ for all } \mathcal{X}, \mathcal{Y} \subseteq Sb(U),\\ &\left(\bigcup\mathcal{X}\right)^\smile = \bigcup\{\breve{X} : X \in \mathcal{X}\} \text{ for all } \mathcal{X} \subseteq Sb(U). \end{aligned}$$

- *If* (C) *and* (I) *then S is an involution, i.e., $S : U \to U$ and $S(S(x)) = x$ for all $x \in U$,*
- *$\mathfrak{Cm}T$ is a nonassociative relation algebra iff* (C) *and* (I),
- *$\mathfrak{Cm}T$ is a relation algebra iff* (C), (I), *and* (A),

Condition (A) is illustrated in Figure 1.

3. A general construction

Let $\mathfrak{A} = \langle A, +, -, ;, {}^\smile, 1' \rangle$ be a complete atomic weakly associative relation algebra [6, 1.2]. (Finite algebras are complete and atomic.) Let n be a positive integer. An $n \times n$ **matrix** is a function defined on $\{1, \ldots, n\}$. The arguments to a matrix are written as subscripts, and the matrix itself is displayed in the usual manner:

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

For each positive integer n, let $M_n\mathfrak{A}$ be the set of $n \times n$ matrices a of atoms of $\mathfrak{A}$ such that if $1 \leq i, j, k \leq n$ then $a_{ii} \leq 1'$, $(a_{ij})^\smile = a_{ji}$, and $a_{ik} \leq a_{ij} ; a_{jk}$. Whenever $a \in M_n\mathfrak{A}$ and $1 \leq i \leq n$ define $\partial_i a$ to be the $(n-1) \times (n-1)$ matrix obtained from a by deleting row i and column i from a, that is, if $1 \leq j, k \leq n-1$, then

$$(\partial_i a)_{j,k} = \begin{cases} a_{j,k} & \text{if } j, k < i \\ a_{j,k+1} & \text{if } j < i \leq k \\ a_{j+1,k} & \text{if } k < i \leq j \\ a_{j+1,k+1} & \text{if } i \leq j, k \end{cases}$$

Whenever $1 \leq i < j \leq n$, let $\partial_{i,j} a := \partial_i \partial_j a = \partial_{j-1} \partial_i a$. For any $X \subseteq M_n\mathfrak{A}$, let $\partial_i X := \{\partial_i a : a \in X\}$. If σ is any permutation of $\{1, \ldots, n\}$ and a is any matrix of elements of $\mathfrak{A}$, let $a\sigma$ be the matrix defined by $(a\sigma)_{ij} = a_{\sigma(i),\sigma(j)}$ whenever $1 \leq i, j \leq n$. Note that for any $n \times n$ matrix a and any permutation σ of n, $a\sigma \in M_n\mathfrak{A}$ iff $a \in M_n\mathfrak{A}$. For every element $x \in A$, the *domain of* x is $x^{\mathsf{d}} = 1' \cdot x ; \breve{x}$, and the *range of* x is $x^{\mathsf{r}} = 1' \cdot \breve{x} ; x$. If a is an atom of $\mathfrak{A}$ then a^{d}, a^{r}, and $\breve{a}$ are also atoms of $\mathfrak{A}$ [6, 3.4, 3.5]. Also, $(a_{ij})^{\mathsf{d}} = a_{ii}$ and $(a_{ij})^{\mathsf{r}} = a_{jj}$ for all $a \in M_n\mathfrak{A}$ and $i, j < n$. Consider a typical member of $M_4\mathfrak{A}$:

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \in M_4\mathfrak{A}.$$

Think of a_{ij} as a binary relation that relates object number i with with object number j. Then the 4×4 matrix a is a list of 16 relations among 4 objects. Some of those relations are deducible from others. For example, the entries on and below the main diagonal of a can be computed from the remaining six entries that lie above the main diagonal: $a_{11} = (a_{12})^{\mathsf{d}}$, $a_{21} = (a_{12})^\smile$, $a_{31} = (a_{13})^\smile$, and so on. The redundant entries may be omitted from a, and the remaining ones illustrated by arrows between the digits 1,$\cdots$,4, each arrow labelled with an appropriate entry from a, as follows:

$$a = \begin{bmatrix} (a_{12})^{\mathsf{d}} & a_{12} & a_{13} & a_{14} \\ (a_{12})^\smile & (a_{12})^{\mathsf{r}} & a_{23} & a_{24} \\ (a_{13})^\smile & (a_{23})^\smile & (a_{34})^{\mathsf{d}} & a_{34} \\ (a_{14})^\smile & (a_{14})^\smile & (a_{34})^\smile & (a_{34})^{\mathsf{r}} \end{bmatrix} = \begin{bmatrix} \cdot & a_{12} & a_{13} & a_{14} \\ \cdot & \cdot & a_{23} & a_{24} \\ \cdot & \cdot & \cdot & a_{34} \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{tikzcd} 1 \arrow[r, "a_{13}"] \arrow[d, "a_{12}"] \arrow[dr, "a_{14}"] & 3 \arrow[d, "a_{34}"] \\ 2 \arrow[r, "a_{24}"] \arrow[ur, "a_{23}"] & 4 \end{tikzcd}.$$

Define a ternary relation $T_2(\mathfrak{A})$ on $M_4\mathfrak{A}$: a triple $\langle a, b, c \rangle$ is in $T_2(\mathfrak{A})$ if $a, b, c \in M_4\mathfrak{A}$ and there is some $m \in M_6\mathfrak{A}$ such that

$$m = \left[\begin{array}{cc|cc|cc} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ \hline m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{array}\right] = \left[\begin{array}{c|c|c} \bar{m}_{11} & \bar{m}_{12} & \bar{m}_{13} \\ \hline \bar{m}_{21} & \bar{m}_{22} & \bar{m}_{23} \\ \hline \bar{m}_{31} & \bar{m}_{32} & \bar{m}_{33} \end{array}\right],$$

$$a = \partial_{5,6}m = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \left[\begin{array}{c|c} \bar{m}_{11} & \bar{m}_{12} \\ \hline \bar{m}_{21} & \bar{m}_{22} \end{array}\right] = \text{ the upper left } 4/9 \text{ of } m,$$

$$b = \partial_{1,2}m = \begin{bmatrix} m_{33} & m_{34} & m_{35} & m_{36} \\ m_{43} & m_{44} & m_{45} & m_{46} \\ m_{53} & m_{54} & m_{55} & m_{56} \\ m_{63} & m_{64} & m_{65} & m_{66} \end{bmatrix} = \left[\begin{array}{c|c} \bar{m}_{22} & \bar{m}_{23} \\ \hline \bar{m}_{32} & \bar{m}_{33} \end{array}\right] = \text{ the lower right } 4/9 \text{ of } m,$$

$$c = \partial_{3,4}m = \begin{bmatrix} m_{11} & m_{12} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{25} & m_{26} \\ m_{51} & m_{52} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{65} & m_{66} \end{bmatrix} = \left[\begin{array}{c|c} \bar{m}_{11} & \bar{m}_{13} \\ \hline \bar{m}_{31} & \bar{m}_{33} \end{array}\right] = \text{ the four } 2 \times 2 \text{ corners of } m.$$

In brief,

$$T_2(\mathfrak{A}) = \{\langle \partial_{5,6}m, \partial_{1,2}m, \partial_{3,4}m \rangle : m \in M_6\mathfrak{A}\}.$$

The ternary relation $T_2(\mathfrak{A})$ determines a complex algebra $\mathfrak{Cm}(T_2(\mathfrak{A}))$ with universe $\mathit{Sb}\,(M_4\mathfrak{A})$. In this algebra, the product $\{a\};\{b\}$ is nonempty only if

$$\partial_{1,2}a = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \partial_{3,4}b$$

(or, equivalently, $a_{34} = b_{12}$), in which case we have this situation:

$$\left[\begin{array}{cc|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdot & \cdot \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & b_{13} & b_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} & b_{23} & b_{24} \\ \hline \cdot & \cdot & b_{31} & b_{32} & b_{33} & b_{34} \\ \cdot & \cdot & b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right] = \left[\begin{array}{cc|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdot & \cdot \\ \hline a_{31} & a_{32} & b_{11} & b_{12} & b_{13} & b_{14} \\ a_{41} & a_{42} & b_{21} & b_{22} & b_{23} & b_{24} \\ \hline \cdot & \cdot & b_{31} & b_{32} & b_{33} & b_{34} \\ \cdot & \cdot & b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right]$$

$$= \begin{tikzcd} 1 \arrow[r, "a_{13}"] \arrow[d, "a_{12}"] \arrow[dr, "a_{14}" near end] & 3 \arrow[r, "b_{13}"] \arrow[d, "a_{34} = b_{12}"] \arrow[dr, "b_{14}" near end] & 5 \arrow[d, "b_{24}"] \\ 2 \arrow[r, "a_{24}"] \arrow[ur, "a_{23}" near start] & 4 \arrow[r, "b_{24}"] \arrow[ur, "b_{23}" near start] & 6 \end{tikzcd} .$$

Computing the product $\{a\};\{b\}$ reduces to finding all the ways that the partial matrix can be completed to get one in $M_6\mathfrak{A}$. The missing entries are the ones that would be labels on arrows from 1 and 2 to 5 and 6 in the illustration. Let U, S and I be defined by (1), (2), and (3), respectively, with $T = T_2(\mathfrak{A})$. For each $x \in \mathit{At}\mathfrak{A}$ let

$$1\text{'}_x = \begin{bmatrix} x^{\mathsf{d}} & x & x^{\mathsf{d}} & x \\ \breve{x} & x^{\mathsf{r}} & \breve{x} & x^{\mathsf{r}} \\ x^{\mathsf{d}} & x & x^{\mathsf{d}} & x \\ \breve{x} & x^{\mathsf{r}} & \breve{x} & x^{\mathsf{r}} \end{bmatrix} = \begin{bmatrix} 1\text{'} \cdot x;\breve{x} & x & 1\text{'} \cdot x;\breve{x} & x \\ \breve{x} & 1\text{'} \cdot \breve{x};x & \breve{x} & 1\text{'} \cdot \breve{x};x \\ 1\text{'} \cdot x;\breve{x} & x & 1\text{'} \cdot x;\breve{x} & x \\ \breve{x} & 1\text{'} \cdot \breve{x};x & \breve{x} & 1\text{'} \cdot \breve{x};x \end{bmatrix} = \begin{tikzcd} 1 \arrow[r, equals, "x^{\mathsf{d}}"] \arrow[d, "x"] \arrow[dr, "x" near end] & 3 \arrow[d, "x"] \\ 2 \arrow[r, equals, "x^{\mathsf{r}}"] \arrow[ur, "\breve{x}" near start] & 4 \end{tikzcd}$$

and note that $1\text{'}_x \in M_4\mathfrak{A}$ for every atom x. Hence $U = M_4\mathfrak{A}$. Next we show that $I = \{1\text{'}_x : x \in \mathit{At}\mathfrak{A}\}$. Suppose

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \in I.$$

Let

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} \end{bmatrix}.$$

Note that $\partial_{5,6}m = a$ and $m \in M_6\mathfrak{A}$ so $\langle a, \partial_{1,2}m, \partial_{3,4}m\rangle \in T_2(\mathfrak{A})$, hence $\partial_{1,2}m = \partial_{3,4}m$ since $a \in I$, that is,

$$\partial_{1,2}m = \begin{bmatrix} a_{33} & a_{34} & a_{31} & a_{32} \\ a_{43} & a_{44} & a_{41} & a_{42} \\ a_{13} & a_{14} & a_{11} & a_{12} \\ a_{23} & a_{24} & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{bmatrix} = \partial_{3,4}m,$$

which implies

$$\begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

so

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (a_{12})^{\mathsf{d}} & a_{12} & (a_{12})^{\mathsf{d}} & a_{12} \\ (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} \\ (a_{12})^{\mathsf{d}} & a_{12} & (a_{12})^{\mathsf{d}} & a_{12} \\ (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} \end{bmatrix} = 1\text{'}_{a_{12}}.$$

For the converse, suppose that $\langle 1\text{'}_x, a, b\rangle \in T_2(\mathfrak{A})$. Then $\partial_{5,6}m = 1\text{'}_x$, $\partial_{1,2}m = a$, and $\partial_{3,4}m = b$ for some $m \in M_6\mathfrak{A}$, that is,

$$m = \begin{bmatrix} x^{\mathsf{d}} & x & x^{\mathsf{d}} & x & m_{15} & m_{16} \\ \breve{x} & x^{\mathsf{r}} & \breve{x} & x^{\mathsf{r}} & m_{25} & m_{26} \\ x^{\mathsf{d}} & x & x^{\mathsf{d}} & x & m_{35} & m_{36} \\ \breve{x} & x^{\mathsf{r}} & \breve{x} & x^{\mathsf{r}} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{36} \end{bmatrix},$$

$$a = \partial_{1,2}m = \begin{bmatrix} x^{\mathsf{d}} & x & m_{35} & m_{36} \\ \breve{x} & x^{\mathsf{r}} & m_{45} & m_{46} \\ m_{53} & m_{54} & m_{55} & m_{56} \\ m_{63} & m_{64} & m_{65} & m_{66} \end{bmatrix},$$

$$b = \partial_{3,4}m = \begin{bmatrix} x^{\mathsf{d}} & x & m_{15} & m_{16} \\ \breve{x} & x^{\mathsf{r}} & m_{25} & m_{26} \\ m_{51} & m_{52} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{65} & m_{66} \end{bmatrix}.$$

Simple computations show that $a = b$. For example,

$$a_{13} = (\partial_{1,2}m)_{13} = m_{35} \leq m_{31} ; m_{15} = x^{\mathsf{d}} ; m_{15} \leq 1\text{'} ; m_{15} = m_{15} = (\partial_{3,4}m)_{13} = b_{13},$$

so $a_{13} = b_{13}$ since a_{13} and b_{13} are atoms. Similarly, if $\langle a, 1\text{'}_x, b\rangle \in T_2(\mathfrak{A})$ then $a = b$. Next, show (I) holds. Let

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \in M_4\mathfrak{A}.$$

Then

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (a_{12})^{\mathsf{d}} & a_{12} \\ (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} \end{bmatrix}$$

so a and $1'_{a_{12}}$ fit together to make the partial matrix

$$\begin{bmatrix} (a_{12})^{\mathsf{d}} & a_{12} & (a_{12})^{\mathsf{d}} & a_{12} & & \\ (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & & \\ (a_{12})^{\mathsf{d}} & a_{12} & (a_{12})^{\mathsf{d}} & a_{12} & a_{13} & a_{14} \\ (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & a_{23} & a_{24} \\ & & a_{31} & a_{32} & a_{33} & a_{34} \\ & & a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

There is exactly one way to complete this partial matrix to make one that is in $M_6\mathfrak{A}$, namely,

$$m = \begin{bmatrix} (a_{12})^{\mathsf{d}} & a_{12} & (a_{12})^{\mathsf{d}} & a_{12} & a_{13} & a_{14} \\ (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & a_{23} & a_{24} \\ (a_{12})^{\mathsf{d}} & a_{12} & (a_{12})^{\mathsf{d}} & a_{12} & a_{13} & a_{14} \\ (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & (a_{12})^{\smile} & (a_{12})^{\mathsf{r}} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

From this we see that $\partial_{1,2}m = \partial_{3,4}m = a$ and $\partial_{5,6}m = 1'_{a_{12}}$, hence $\langle 1'_{a_{12}}, a, a\rangle \in T_2(\mathfrak{A})$ and, similarly, $\langle a, 1'_{a_{12}}, a\rangle \in T_2(\mathfrak{A})$. It follows that (I) holds.

To prove (C) it suffices to show, assuming $a \in M_4\mathfrak{A}$, that $\langle a, a(13)(24)\rangle \in S$. Let $x, y \in M_4\mathfrak{A}$, and suppose $\langle a, x, y\rangle \in T_2(\mathfrak{A})$. Then, for some $m \in M_6\mathfrak{A}$, $\partial_{5,6}m = a$, $\partial_{1,2}m = x$, and $\partial_{3,4}m = y$. Hence

$$m(13)(24) = \begin{bmatrix} m_{33} & m_{34} & m_{31} & m_{32} & m_{35} & m_{36} \\ m_{43} & m_{44} & m_{41} & m_{42} & m_{45} & m_{46} \\ m_{13} & m_{14} & m_{11} & m_{12} & m_{15} & m_{16} \\ m_{23} & m_{24} & m_{21} & m_{22} & m_{25} & m_{26} \\ m_{53} & m_{54} & m_{51} & m_{52} & m_{55} & m_{56} \\ m_{63} & m_{64} & m_{61} & m_{62} & m_{65} & m_{66} \end{bmatrix} \in M_6\mathfrak{A}$$

$$\begin{aligned} \partial_{5,6}\left(m(13)(24)\right) &= \left(\partial_{5,6}m\right)(13)(24) = a(13)(24), \\ \partial_{1,2}\left(m(13)(24)\right) &= \partial_{3,4}m = y, \\ \partial_{3,4}\left(m(13)(24)\right) &= \partial_{1,2}m = x, \end{aligned}$$

so $\langle a(13)(24), y, x\rangle \in T_2(\mathfrak{A})$. Since (C) and (I) hold, S is an involution, so $\breve{X} = \{a(13)(24) : a \in X\}$ for all $X \subseteq M_4\mathfrak{A}$. The only reason $\mathfrak{Cm}(T_2(\mathfrak{A}))$ may not be a relation algebra is that the associative law for relative multiplication may fail.

Theorem 2. *If $\mathfrak{A}$ is a complete atomic weakly associative relation algebra, then $\mathfrak{Cm}(T_2(\mathfrak{A}))$ is a complete atomic nonassociative relation algebra.*

The algebra $\mathfrak{Cm}(T_2(\mathfrak{A}))$ need not be a relation algebra, but it does turn out to be a relation algebra for certain choices of $\mathfrak{A}$.

4. Examples yielding relation algebras

In this section we define a finite algebra $\mathfrak{A}_{n,\beta}$ whenever $0 < n < \beta < \omega$. The construction used here is taken from [1], but the algebras from [8] can also be used. Let

$$U_{n,\beta} = \{1', e, p_1, \ldots, p_n, q_0, \ldots, q_\beta\}.$$

A triple $\langle a, b, c\rangle$ is said to be **forbidden** if one of the following conditions holds:

$$\begin{aligned} &a = 1' \text{ and } b \neq c, \\ &b = 1' \text{ and } a \neq c, \\ &c = 1' \text{ and } a \neq b, \end{aligned}$$

$$\begin{aligned}
&\{a,b,c\} \subseteq \{e,q_0\},\\
&\{a,b,c\} \subseteq \{q_i : i \le \beta\},\\
&\{a,b,c\} \subseteq \{p_i,q_i\} \text{ for some } i \text{ with } 1 \le i \le n,\\
&\{a,b,c\} = \{e,q_i,q_j\} \text{ for some } i,j \le \beta \text{ with } |i-j| > 1.
\end{aligned}$$

Any triple that is not forbidden is said to be **mandatory**. Let $C_{n,\beta}$ be the set of mandatory triples, and let $\mathfrak{A}_{n,\beta} = \mathfrak{Cm}\,(C_{n,\beta})$. Note that $\mathfrak{A}_{n,\beta}$ is generated by e without using 1', since

$$\begin{aligned}
q_0 &= \overline{e\,;e} \cdot \overline{e},\\
1' &= q_0\,;q_0 \cdot \overline{e\,;q_0},\\
q_1 &= e\,;q_0 \cdot \overline{q_0\,;q_0},\\
q_2 &= e\,;q_1 \cdot \overline{e\,;q_0} \cdot \overline{e+q_0},\\
q_i &= e\,;q_{i-1} \cdot \overline{e\,;q_{i-2}} && \text{for } 3 \le i \le \beta,\\
p_i &= q_0\,;q_0 \cdot \overline{q_i\,;q_i} && \text{for } 1 \le i \le n.
\end{aligned}$$

As shown in the next theorem, $\mathfrak{Cm}\,(T_2\,(\mathfrak{A}_{n,\beta}))$ is a relation algebra whenever $n > 6$. If β is sufficiently large compared to n, then $\mathfrak{A}_{n,\beta}$ is not representable [1]. This nonrepresentability is inherited by $\mathfrak{Cm}\,(T_2\,(\mathfrak{A}_{n,\beta}))$, because the original algebra can be recovered from the "doubling" construction. Since $\mathfrak{A}_{n,\beta}$ is generated by a single element, namely e, it follows, as we shall see in the final section, that $\mathfrak{Cm}\,(T_2\,(\mathfrak{A}_{n,\beta}))$ has a nonrepresentable subalgebra generated by two functional elements.

Theorem 3. *If $6 < n < \beta < \omega$ then $\mathfrak{Cm}\,(T_2\,(\mathfrak{A}_{n,\beta}))$ is a relation algebra.*

Proof. By Theorem 2, $\mathfrak{Cm}\,(T_2\,(\mathfrak{A}_{n,\beta}))$ is a relation algebra if it satisfies the associative law for relative multiplication. Assume the hypothesis of (A), namely $\langle x,y,z\rangle, \langle z,a,b\rangle \in T$. There are $m, m' \in M_6\mathfrak{A}$ such that $x = \partial_{5,6}m$, $y = \partial_{1,2}m$, $z = \partial_{3,4}m = \partial_{1,2}(m(13)(24)) = \partial_{5,6}m'$, $a = \partial_{1,2}m'$, and $b = \partial_{3,4}m'$, so $m(13)(24)$ and m' fit together to make a partial matrix

$$\begin{bmatrix}
m_{33} & m_{34} & m_{31} & m_{32} & m_{35} & m_{36} & \cdot & \cdot\\
m_{43} & m_{44} & m_{41} & m_{42} & m_{45} & m_{46} & \cdot & \cdot\\
m_{13} & m_{14} & m_{11} & m_{12} & m_{15} & m_{16} & m'_{15} & m'_{16}\\
m_{23} & m_{24} & m_{21} & m_{22} & m_{25} & m_{26} & m'_{25} & m'_{26}\\
m_{53} & m_{54} & m_{51} & m_{52} & m_{55} & m_{56} & m'_{35} & m'_{36}\\
m_{63} & m_{64} & m_{61} & m_{62} & m_{65} & m_{66} & m'_{45} & m'_{46}\\
\cdot & \cdot & m'_{51} & m'_{52} & m'_{53} & m'_{54} & m'_{55} & m'_{56}\\
\cdot & \cdot & m'_{61} & m'_{62} & m'_{63} & m'_{64} & m'_{65} & m'_{66}
\end{bmatrix}$$

$$= \begin{bmatrix}
m_{33} & m_{34} & m_{31} & m_{32} & m_{35} & m_{36} & \cdot & \cdot\\
m_{43} & m_{44} & m_{41} & m_{42} & m_{45} & m_{46} & \cdot & \cdot\\
m_{13} & m_{14} & m'_{11} & m'_{12} & m'_{13} & m'_{14} & m'_{15} & m'_{16}\\
m_{23} & m_{24} & m'_{21} & m'_{22} & m'_{23} & m'_{24} & m'_{25} & m'_{26}\\
m_{53} & m_{54} & m'_{31} & m'_{32} & m'_{33} & m'_{34} & m'_{35} & m'_{36}\\
m_{63} & m_{64} & m'_{41} & m'_{42} & m'_{43} & m'_{44} & m'_{45} & m'_{46}\\
\cdot & \cdot & m'_{51} & m'_{52} & m'_{53} & m'_{54} & m'_{55} & m'_{56}\\
\cdot & \cdot & m'_{61} & m'_{62} & m'_{63} & m'_{64} & m'_{65} & m'_{66}
\end{bmatrix}.$$

Suppose $m'_{56} \le 0'$. (The case $m'_{56} \le 1'$ is similar but simpler.) Choose i so that

$$\{p_i,q_i\} \cap \{m_{34}, m_{31}, m_{32}, m_{35}, m_{36}, m'_{56}\} = \emptyset,$$

and choose j so that

$$\{p_j,q_j\} \cap \{m_{34}, m_{41}, m_{42}, m_{45}, m_{46}, m'_{56}\} = \emptyset.$$

This is always possible if $n > 6$. Let

$$m'' = \begin{bmatrix} m_{33} & m_{34} & m_{31} & m_{32} & m_{35} & m_{36} & p_i & p_i \\ m_{43} & m_{44} & m_{41} & m_{42} & m_{45} & m_{46} & p_j & p_j \\ m_{13} & m_{14} & m_{11} & m_{12} & m_{15} & m_{16} & m'_{15} & m'_{16} \\ m_{23} & m_{24} & m_{21} & m_{22} & m_{25} & m_{26} & m'_{25} & m'_{26} \\ m_{53} & m_{54} & m_{51} & m_{52} & m_{55} & m_{56} & m'_{35} & m'_{36} \\ m_{63} & m_{64} & m_{61} & m_{62} & m_{65} & m_{66} & m'_{45} & m'_{46} \\ p_i & p_j & m'_{51} & m'_{52} & m'_{53} & m'_{54} & m'_{55} & m'_{56} \\ p_i & p_j & m'_{61} & m'_{62} & m'_{63} & m'_{64} & m'_{65} & m'_{66} \end{bmatrix}$$

It is straightforward to check that $m'' \in M_8\mathfrak{A}$. Set

$$m''' = (\partial_{5,6} m'')\,(13)(24) = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m'_{15} & m'_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m'_{25} & m'_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & p_i & p_i \\ m_{41} & m_{42} & m_{43} & m_{44} & p_j & p_j \\ m'_{51} & m'_{52} & p_i & p_j & m'_{55} & m'_{56} \\ m'_{61} & m'_{62} & p_i & p_j & m'_{65} & m'_{66} \end{bmatrix}$$

and

$$m'''' = \partial_{3,4} m'' = \begin{bmatrix} m_{33} & m_{34} & m_{35} & m_{36} & p_i & p_i \\ m_{43} & m_{44} & m_{45} & m_{46} & p_j & p_j \\ m_{53} & m_{54} & m_{55} & m_{56} & m'_{35} & m'_{36} \\ m_{63} & m_{64} & m_{65} & m_{66} & m'_{45} & m'_{46} \\ p_i & p_j & m'_{53} & m'_{54} & m'_{55} & m'_{56} \\ p_i & p_j & m'_{63} & m'_{64} & m'_{65} & m'_{66} \end{bmatrix}.$$

Then $m''', m'''' \in M_6\mathfrak{A}$. Let $c = \partial_{1,2} m'''$. Then

$$\langle \partial_{5,6} m''', \partial_{1,2} m''', \partial_{3,4} m''' \rangle = \langle x, c, b \rangle \in T_2\left(\mathfrak{A}_{n,\beta}\right)$$

and

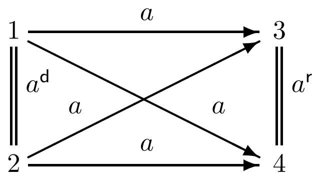
$$\langle \partial_{5,6} m'''', \partial_{1,2} m'''', \partial_{3,4} m'''' \rangle = \langle y, a, c \rangle \in T_2\left(\mathfrak{A}_{n,\beta}\right),$$

so (A) holds. □

5. RECOVERING THE ORIGINAL ALGEBRA

Again consider an arbitrary complete and atomic weakly associative relation algebra $\mathfrak{A}$. For each atom $a \in At\mathfrak{A}$ let

$$\iota(a) = \begin{bmatrix} a^{\mathsf{d}} & a^{\mathsf{d}} & a & a \\ a^{\mathsf{d}} & a^{\mathsf{d}} & a & a \\ \breve{a} & \breve{a} & a^{\mathsf{r}} & a^{\mathsf{r}} \\ \breve{a} & \breve{a} & a^{\mathsf{r}} & a^{\mathsf{r}} \end{bmatrix} =$$



and note that $\iota(a) \in M_4\mathfrak{A}$.

Theorem 4. *Assume $\mathfrak{A}$ is a complete and atomic weakly associative relation algebra.*

1. *For any atoms a, b, c of $\mathfrak{A}$, $a;b \geq c$ iff $\langle \iota(a), \iota(b), \iota(c) \rangle \in T_2(\mathfrak{A})$.*
2. *If $a, b \in At\mathfrak{A}$ then, in $\mathfrak{Cm}\,(T_2(\mathfrak{A}))$, $\{\iota(a)\};\{\iota(b)\} = \{\iota(c) : a;b \geq c \in At\mathfrak{A}\}$.*
3. *$\{\iota(a) : a \in At\mathfrak{A}\}$ is an equivalence element in $\mathfrak{Cm}(T_2(\mathfrak{A}))$.*
4. *$\mathfrak{A}$ is isomorphic to the relativization of $\mathfrak{Cm}(T_2(\mathfrak{A}))$ to $\{\iota(a) : a \in At\mathfrak{A}\}$ via φ, where*

$$\varphi(x) = \{\iota(a) : x \geq a \in At\mathfrak{A}\}.$$

5. *If $\mathfrak{Cm}(T_2(\mathfrak{A}))$ is representable, then $\mathfrak{A}$ is representable.*

Proof. Let a, b, c be atoms. Suppose $\langle \iota(a), \iota(b), \iota(c)\rangle \in T_2(\mathfrak{A})$. Then there is some $m \in M_6\mathfrak{A}$ such that

$$\langle \partial_{5,6}m, \partial_{1,2}m, \partial_{3,4}m\rangle = \langle \iota(a), \iota(b), \iota(c)\rangle\,,$$

so

$$a;b = \iota(a)_{13};\iota(b)_{13} = (\partial_{5,6}m)_{13};(\partial_{1,2}m)_{13} = m_{13};m_{35} \geq m_{15} = (\partial_{3,4}m)_{13} = \iota(c)_{13} = c.$$

Conversely, if $a;b \geq c$, then $a^{\mathsf{r}} = b^{\mathsf{d}}$, $a^{\mathsf{d}} = c^{\mathsf{d}}$, and $b^{\mathsf{r}} = c^{\mathsf{r}}$ [6, 5.12], and

$$m = \begin{bmatrix} a^{\mathsf{d}} & a^{\mathsf{d}} & a & a & c & c \\ a^{\mathsf{d}} & a^{\mathsf{d}} & a & a & c & c \\ \breve{a} & \breve{a} & b^{\mathsf{d}} & b^{\mathsf{d}} & b & b \\ \breve{a} & \breve{a} & b^{\mathsf{d}} & b^{\mathsf{d}} & b & b \\ \breve{c} & \breve{c} & \breve{b} & \breve{b} & c^{\mathsf{r}} & c^{\mathsf{r}} \\ \breve{c} & \breve{c} & \breve{b} & \breve{b} & c^{\mathsf{r}} & c^{\mathsf{r}} \end{bmatrix} \in M_6\mathfrak{A}$$

(all the required conditions are easily derivable from $a;b \geq c$ and the assumption that a, b, c are atoms). For part (2), suppose a and b are atoms. If c is an atom such that $c \leq a;b$ then, from what was shown above, we have $\langle \iota(a), \iota(b), \iota(c)\rangle \in T_2(\mathfrak{A})$, so $\iota(c) \in \{\iota(a)\};\{\iota(b)\}$. For the inclusion in the other direction, suppose $m \in M_6\mathfrak{A}$, $\partial_{5,6}m = \iota(a)$, and $\partial_{1,2}m = \iota(b)$. Let $c = m_{15}$. Then

$$c = m_{15} \leq m_{13};m_{35} = (\partial_{5,6}m)_{13};(\partial_{1,2}m)_{13} = \iota(a)_{13};\iota(b)_{13} = a;b,$$

so $a^{\mathsf{d}} = c^{\mathsf{d}}$, $a^{\mathsf{r}} = b^{\mathsf{d}}$, and $b^{\mathsf{r}} = c^{\mathsf{r}}$. Hence

$$m_{25} \leq m_{21};m_{15} = (\partial_{5,6}m)_{21};c = \iota(a)_{21};c = a^{\mathsf{d}};c = c^{\mathsf{d}};c = c,$$

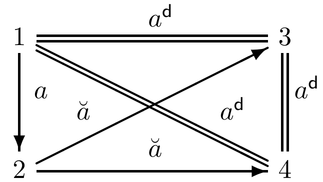
and, similarly, $c = m_{16} = m_{26}$. In addition, $\breve{c} = m_{51} = m_{52} = m_{61} = m_{62}$. Therefore, $\partial_{3,4}m = \iota(c)$.

Part (5) follows from the fact that the relativization of a representable relation algebra to an equivalence element is representable [2]. □

6. Functional elements

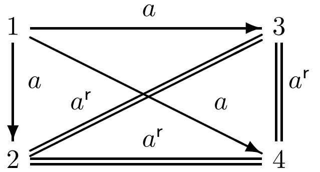
For every atom $a \in At\mathfrak{A}$, let

$$\pi(a) = \begin{bmatrix} a^{\mathsf{d}} & a & a^{\mathsf{d}} & a^{\mathsf{d}} \\ \breve{a} & a^{\mathsf{r}} & \breve{a} & \breve{a} \\ a^{\mathsf{d}} & a & a^{\mathsf{d}} & a^{\mathsf{d}} \\ a^{\mathsf{d}} & a & a^{\mathsf{d}} & a^{\mathsf{d}} \end{bmatrix} =$$



,

$$\rho(a) = \begin{bmatrix} a^{\mathsf{d}} & a & a & a \\ \breve{a} & a^{\mathsf{r}} & a^{\mathsf{r}} & a^{\mathsf{r}} \\ \breve{a} & a^{\mathsf{r}} & a^{\mathsf{r}} & a^{\mathsf{r}} \\ \breve{a} & a^{\mathsf{r}} & a^{\mathsf{r}} & a^{\mathsf{r}} \end{bmatrix} =$$



.

Then $\{\pi(a)\}$ and $\{\rho(a)\}$ are functional elements of $\mathfrak{Cm}(T_2(\mathfrak{A}))$, and $\iota(a)$ can be recovered from $\pi(a)$ and $\rho(a)$. In fact,

$$(\{\pi(a)\})\breve{\ };\{\pi(a)\} = (\{\rho(a)\})\breve{\ };\{\rho(a)\} = \{\iota(a^{\mathsf{d}})\} = \{1'_{a^{\mathsf{d}}}\},$$
$$(\{\pi(a)\})\breve{\ };\{\rho(a)\} = \{\iota(a)\}.$$

Choose any $n > 6$. Then $\mathfrak{Cm}(T_2(\mathfrak{A}_{n,\beta}))$ is a relation algebra. Choose β large enough so that $\mathfrak{A}_{n,\beta}$ is not representable [1]. Then $\mathfrak{Cm}(T_2(\mathfrak{A}_{n,\beta}))$ is also not representable. Recall that $\mathfrak{A}_{n,\beta}$ is generated by one of its atoms, namely e, and $\mathfrak{A}_{n,\beta}$ is integral (1' is an atom), so $e^{\mathsf{d}} = 1'$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{Cm}(T_2(\mathfrak{A}_{n,\beta}))$ generated by the two functional atoms $\{\pi(e)\}$ and $\{\rho(e)\}$. Let

$E = \{\iota(a) : a \in At\mathfrak{A}_{n,\beta}\}$. It follows from the equations above that $\{\iota(e)\}$ and $\{\iota(1\text{'})\}$ are in $\mathfrak{B}$. Recall that the Boolean unit of $\mathfrak{Cm}(T_2(\mathfrak{A}_{n,\beta}))$ is $M_4(\mathfrak{A}_{n,\beta})$, and note that

$$E = \{\iota(1\text{'})\};M_4(\mathfrak{A}_{n,\beta});\{\iota(1\text{'})\}.$$

Thus E is also in $\mathfrak{B}$. The arithmetic of $\mathfrak{A}_{n,\beta}$ can be carried out now in $\mathfrak{B}$:

$$\begin{aligned} \{\iota(q_0)\} &= E \sim (\{\iota(e)\};\{\iota(e)\} \cup \{\iota(e)\})\,, \\ \{\iota(1\text{'})\} &= \{\iota(q_0)\};\{\iota(q_0)\} \sim \{\iota(e)\};\{\iota(q_0)\}, \\ &\vdots \\ &etc. \end{aligned}$$

so $\{\iota(a)\}$ is in $\mathfrak{B}$ for every atom $a \in At\mathfrak{A}_{n,\beta}$. It follows that $\mathfrak{A}_{n,\beta}$ is isomorphic to the relativization of $\mathfrak{B}$ to E. Now $\mathfrak{A}_{n,\beta}$ is not representable, so $\mathfrak{B}$ is not representable and $\mathfrak{B}$ is generated by two functional atoms.

The algebra $\mathfrak{B}$ is finite, but not small. It would be interesting to find smaller examples, and it should be possible to find a smaller example that is also generated by only a single functional atom.

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