Edge-coloring problems

Roger Maddux

January 26, 2009

A classical edge-coloring problem

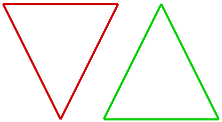
Given: a finite set C of colors and some forbidden triangles (copies of K_3) whose edges are colored with colors in C.

A classical edge-coloring problem

Given: a finite set C of colors and some forbidden triangles (copies of K_3) whose edges are colored with colors in C.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears?

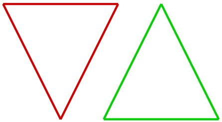
A classical edge-coloring problem

Given: a finite set C of colors and some forbidden triangles (copies of K_3) whose edges are colored with colors in C.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears?
Example: The colors are red and green and the forbidden triangles are the two monochromatic ones:



A classical edge-coloring problem

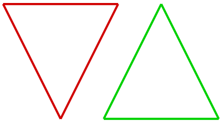
Given: a finite set C of colors and some forbidden triangles (copies of K_3) whose edges are colored with colors in C.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears?
Example: The colors are red and green and the forbidden triangles are the two monochromatic ones:



Theorem: The edges of K_n can be 2-colored so that no monochromatic triangle appears iff $n \in \{1, 2, 3, 4, 5\}$.

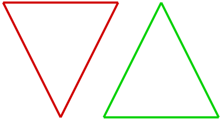
Solution to a classic coloring problem

Proof: The colors are red and green and the forbidden triangles are



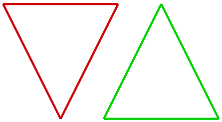
Solution to a classic coloring problem

Proof: The colors are red and green and the forbidden triangles are

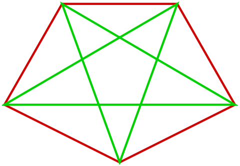


The edges of K_n can be 2-colored so that no monochromatic triangle appears if $n \in \{1, 2, 3, 4, 5\}$. Here's how:

Proof: The colors are red and green and the forbidden triangles are



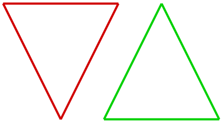
The edges of K_n can be 2-colored so that no monochromatic triangle appears if $n \in \{1, 2, 3, 4, 5\}$. Here's how:



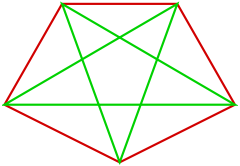
The subgraphs give 2-colorings for K_1, K_2, K_3, K_4.

Solution to a classic coloring problem

Proof: The colors are red and green and the forbidden triangles are



The edges of K_n can be 2-colored so that no monochromatic triangle appears if $n \in \{1, 2, 3, 4, 5\}$. Here's how:



The subgraphs give 2-colorings for K_1, K_2, K_3, K_4.
The Party Theorem: If $n \geq 6$ then every 2-coloring of the edges K_n contains a monochromatic triangle.

A variation on the classic problem

Given: a finite set of colors and some forbidden edge-colored triangles.

A variation on the classic problem

Given: a finite set of colors and some forbidden edge-colored triangles.

Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears and *whatever is not forbidden is mandatory*? (a **proper** coloring)

A variation on the classic problem

Given: a finite set of colors and some forbidden edge-colored triangles.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears and *whatever is not forbidden is mandatory*? (a **proper** coloring)

"Whatever is not forbidden is mandatory" means

A variation on the classic problem

Given: a finite set of colors and some forbidden edge-colored triangles.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears and *whatever is not forbidden is mandatory*? (a **proper** coloring)

"Whatever is not forbidden is mandatory" means every vertex must be part of an edge of each color, and

A variation on the classic problem

Given: a finite set of colors and some forbidden edge-colored triangles.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears and *whatever is not forbidden is mandatory*? (a **proper** coloring)

"Whatever is not forbidden is mandatory" means every vertex must be part of an edge of each color, and if a colored edge could be part of a non-forbidden triangle, then it is.

A variation on the classic problem

Given: a finite set of colors and some forbidden edge-colored triangles.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears and *whatever is not forbidden is mandatory*? (a **proper** coloring)

"Whatever is not forbidden is mandatory" means every vertex must be part of an edge of each color, and if a colored edge could be part of a non-forbidden triangle, then it is.

Definition: If F is a finite set of triangles whose edges have been colored with colors in a finite set C, then the **spectrum** $Sp(C, F)$ is the set of countable cardinals n such that the edges of K_n can be properly colored using colors in C (no triangle in F occurs, and everything mandatory *does* occur). $Sp(C, F) \subseteq \{1, 2, 3, \ldots, \omega\}$.

The previous example, revisited under the new rules

There are two colors, so $C = \{a, b\}$, and the forbidden triangles are the two monochromatic ones, which we'll call *aaa* and *bbb*, so $F = \{aaa, bbb\}$.

The previous example, revisited under the new rules

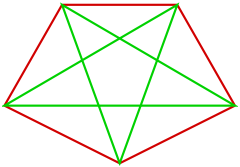
There are two colors, so $C = \{a, b\}$, and the forbidden triangles are the two monochromatic ones, which we'll call *aaa* and *bbb*, so $F = \{aaa, bbb\}$.
Theorem 5.7: $Sp(C, F) = Sp(a, b; aaa, bbb) = \{5\}$.

The previous example, revisited under the new rules

There are two colors, so $C = \{a, b\}$, and the forbidden triangles are the two monochromatic ones, which we'll call *aaa* and *bbb*, so $F = \{aaa, bbb\}$.

Theorem 5.7: $Sp(C, F) = Sp(a, b; aaa, bbb) = \{5\}$.

Proof: The edge-colored K_5 below avoids monochromatic triangles and contains everything mandatory. No subgraph of this graph has everything it needs, and neither does any other 2-coloring of the edges of K_n where $n \leq 4$. (HW: Check this!)



Some general considerations

- There are $\binom{|C|+2}{3}$ triangles with C-colored edges.

For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.

Some general considerations

- There are $\binom{|C|+2}{3}$ triangles with C-colored edges.

For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.

- For *many* sets F of colored triangles, $Sp(C, F) = \emptyset$.

Some general considerations

- There are $\binom{|C|+2}{3}$ triangles with C-colored edges.

For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.

- For *many* sets F of colored triangles, $Sp(C, F) = \emptyset$.

For example, $Sp(a, b; aaa, bbb, baa, abb) = \emptyset$

Some general considerations

- There are $\binom{|C|+2}{3}$ triangles with C-colored edges.

For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.

- For *many* sets F of colored triangles, $Sp(C, F) = \emptyset$.

For example, $Sp(a, b; aaa, bbb, baa, abb) = \emptyset = Sp(a, b; baa, abb)$.

Some general considerations

• There are $\binom{|C|+2}{3}$ triangles with C-colored edges.
For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.
• For *many* sets F of colored triangles, $Sp(C, F) = \emptyset$.
For example, $Sp(a, b; aaa, bbb, baa, abb) = \emptyset = Sp(a, b; baa, abb)$.
• A necessary condition for $Sp(C, F) \neq \emptyset$ is that if $a, b, c, e, d \in C$ and $abc, ced \notin F$ then there is some $f \in C$ such that $aef, fbd \notin F$.

Some general considerations

- There are $\binom{|C|+2}{3}$ triangles with C-colored edges.

For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.

- For *many* sets F of colored triangles, $Sp(C, F) = \emptyset$.

For example, $Sp(a, b; aaa, bbb, baa, abb) = \emptyset = Sp(a, b; baa, abb)$.

- A necessary condition for $Sp(C, F) \neq \emptyset$ is that if $a, b, c, e, d \in C$ and $abc, ced \notin F$ then there is some $f \in C$ such that $aef, fbd \notin F$. Only examples satisfying this condition are considered below.

Some general considerations

- There are $\binom{|C|+2}{3}$ triangles with C-colored edges.

For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.

- For *many* sets F of colored triangles, $Sp(C, F) = \emptyset$.

For example, $Sp(a, b; aaa, bbb, baa, abb) = \emptyset = Sp(a, b; baa, abb)$.

- A necessary condition for $Sp(C, F) \neq \emptyset$ is that if $a, b, c, e, d \in C$ and $abc, ced \notin F$ then there is some $f \in C$ such that $aef, fbd \notin F$. Only examples satisfying this condition are considered below.

- By the Compactness Theorem, if $Sp(C, F)$ contains arbitrarily large integers, then $\omega \in Sp(C, F)$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.
Theorem 2.7: $Sp(a, b; bbb, baa) = \{6, \cdots, \omega\}$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.
Theorem 2.7: $Sp(a, b; bbb, baa) = \{6, \cdots, \omega\}$.
Theorem 3.7: $Sp(a, b; aaa, baa) = \{6, 8, 10, \cdots, \omega\}$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.
Theorem 2.7: $Sp(a, b; bbb, baa) = \{6, \cdots, \omega\}$.
Theorem 3.7: $Sp(a, b; aaa, baa) = \{6, 8, 10, \cdots, \omega\}$.
Theorem 4.7: $Sp(a, b; baa) = \{9, \cdots, \omega\}$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.
Theorem 2.7: $Sp(a, b; bbb, baa) = \{6, \cdots, \omega\}$.
Theorem 3.7: $Sp(a, b; aaa, baa) = \{6, 8, 10, \cdots, \omega\}$.
Theorem 4.7: $Sp(a, b; baa) = \{9, \cdots, \omega\}$.
Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.
Theorem 2.7: $Sp(a, b; bbb, baa) = \{6, \cdots, \omega\}$.
Theorem 3.7: $Sp(a, b; aaa, baa) = \{6, 8, 10, \cdots, \omega\}$.
Theorem 4.7: $Sp(a, b; baa) = \{9, \cdots, \omega\}$.
Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 6.7: $Sp(a, b; bbb) = \{8, \cdots, \omega\}$.

More results on two colors

Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.
Theorem 2.7: $Sp(a, b; bbb, baa) = \{6, \cdots, \omega\}$.
Theorem 3.7: $Sp(a, b; aaa, baa) = \{6, 8, 10, \cdots, \omega\}$.
Theorem 4.7: $Sp(a, b; baa) = \{9, \cdots, \omega\}$.
Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 6.7: $Sp(a, b; bbb) = \{8, \cdots, \omega\}$.
Theorem 7.7: $Sp(a, b;) = \{9, \cdots, \omega\}$. (9: U. Wostner, $\sim$1976)

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55. The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55. The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper. Kramer proved 14 and 15 are impossible.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55. The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper. Kramer proved 14 and 15 are impossible.
Theorem (Ramsey): $\omega \notin Sp(C, \{ccc : c \in C\})$.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55. The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper. Kramer proved 14 and 15 are impossible.
Theorem (Ramsey): $\omega \notin Sp(C, \{ccc : c \in C\})$.
Conjecture: $Sp(C, \{ccc : c \in C\}) \neq \emptyset$.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55. The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper. Kramer proved 14 and 15 are impossible.
Theorem (Ramsey): $\omega \notin Sp(C, \{ccc : c \in C\})$.
Conjecture: $Sp(C, \{ccc : c \in C\}) \neq \emptyset$. True for $|C| = 2$ by Theorem 5.7, for $|C| = 3$ by Theorem 62.65, for $|C| = 4, 5$ by Comer'83.

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55. The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper. Kramer proved 14 and 15 are impossible.
Theorem (Ramsey): $\omega \notin Sp(C, \{ccc : c \in C\})$.
Conjecture: $Sp(C, \{ccc : c \in C\}) \neq \emptyset$. True for $|C| = 2$ by Theorem 5.7, for $|C| = 3$ by Theorem 62.65, for $|C| = 4, 5$ by Comer'83. Erdös, Szemeredi, and Trotter proved it for all sufficiently large finite C,

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$. 17 or more points can't be properly colored by Greenwood-Gleason'55. The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper. Kramer proved 14 and 15 are impossible.
Theorem (Ramsey): $\omega \notin Sp(C, \{ccc : c \in C\})$.
Conjecture: $Sp(C, \{ccc : c \in C\}) \neq \emptyset$. True for $|C| = 2$ by Theorem 5.7, for $|C| = 3$ by Theorem 62.65, for $|C| = 4, 5$ by Comer'83. Erdös, Szemeredi, and Trotter proved it for all sufficiently large finite C, but they were wrong. Try it for 6 colors!

More results on three colors

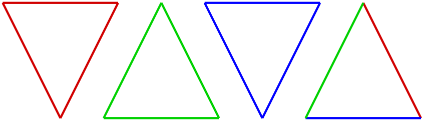
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.

More results on three colors

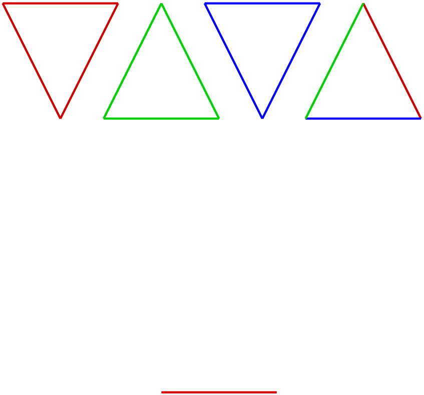
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.

More results on three colors

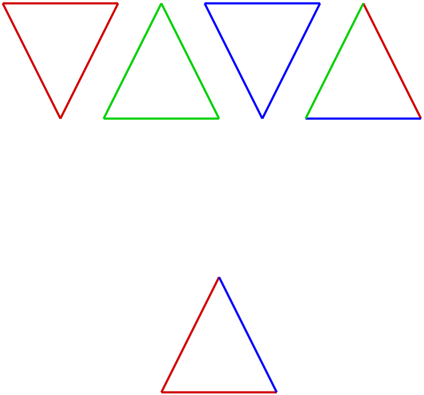
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
A contradiction can be found by looking at only 5 points. If $a =$red, $b =$blue, $c =$green, then the forbidden triangles are



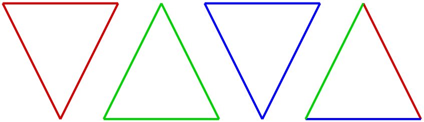
Proof of Theorem 21.65

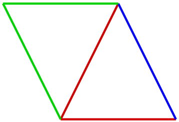


Proof of Theorem 21.65

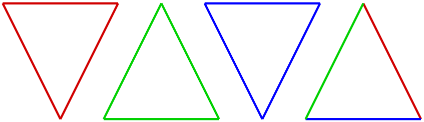


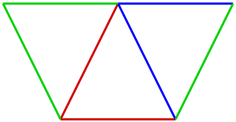
Proof of Theorem 21.65



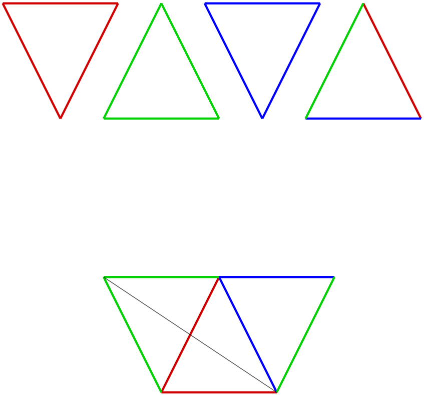


Proof of Theorem 21.65

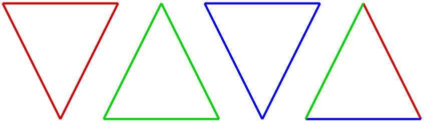


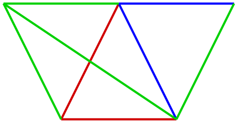


Proof of Theorem 21.65

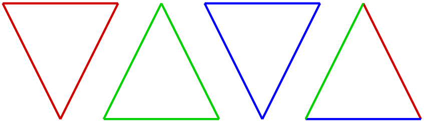


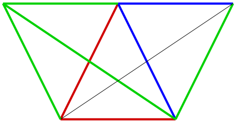
Proof of Theorem 21.65



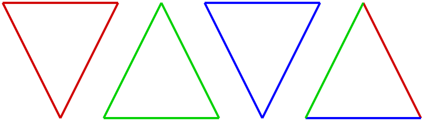


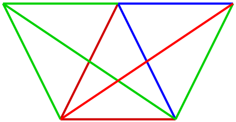
Proof of Theorem 21.65



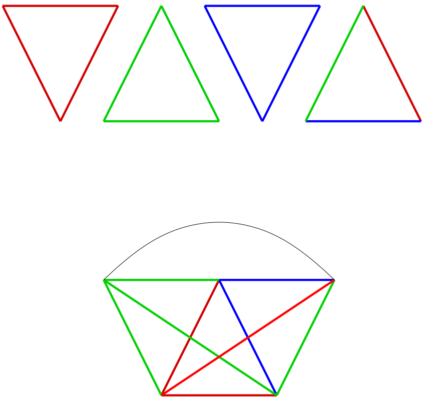


Proof of Theorem 21.65

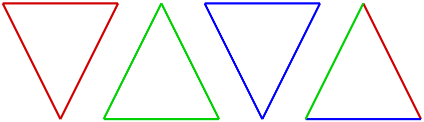


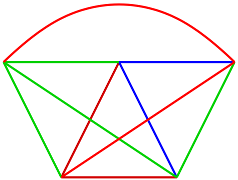


Proof of Theorem 21.65

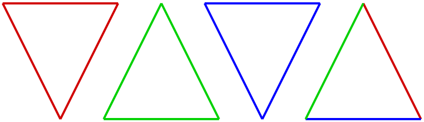


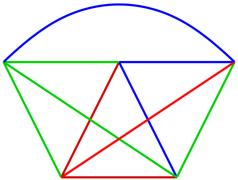
Proof of Theorem 21.65



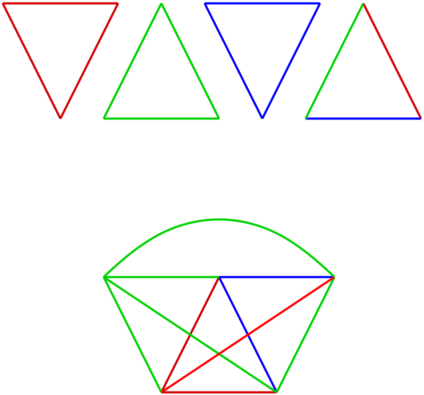


Proof of Theorem 21.65





Proof of Theorem 21.65



More results on three colors

Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.

More results on three colors

Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
Theorem 24.65: $Sp(a, b, c; abc)$

More results on three colors

Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$. (∼1974)

More results on three colors

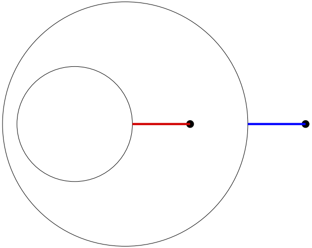
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$. ($\sim$1974)
The proof has two parts: (1) K_ω can be properly colored.

Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$. (∼1974)
The proof has two parts: (1) K_ω can be properly colored.
(2) If K_n is properly colored, then $n \geq \omega$.

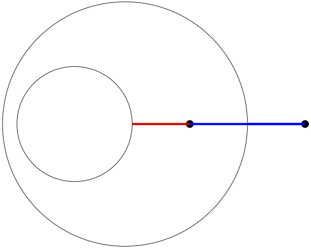
More results on three colors

Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$. (∼1974)
The proof has two parts: (1) K_ω can be properly colored.
(2) If K_n is properly colored, then $n \geq \omega$.
This is a "surprising" result. For example, in 1994 A.Simon conjectured (only briefly, at a meeting) that $\{\omega\}$ cannot be a spectrum.

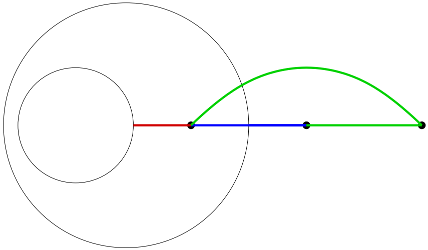
Proof of Theorem 24.65(2)



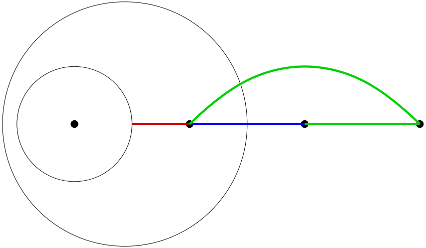
Proof of Theorem 24.65(2)



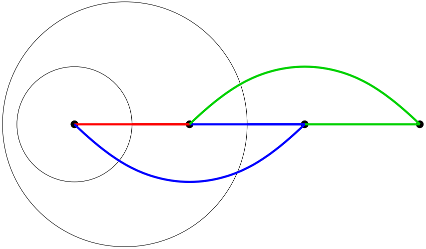
Proof of Theorem 24.65(2)



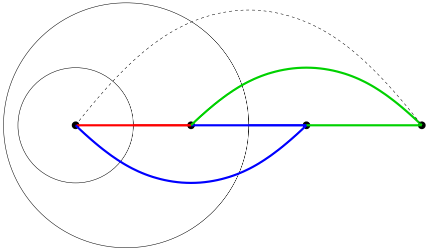
Proof of Theorem 24.65(2)



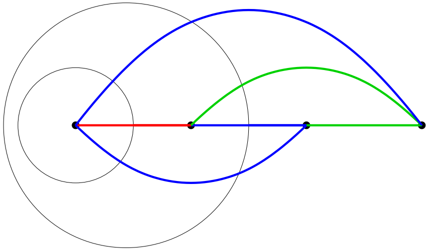
Proof of Theorem 24.65(2)



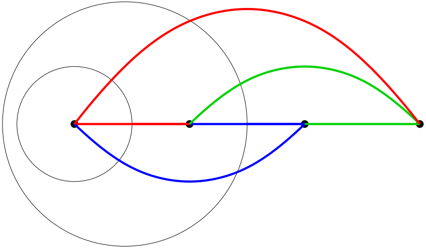
Proof of Theorem 24.65(2)



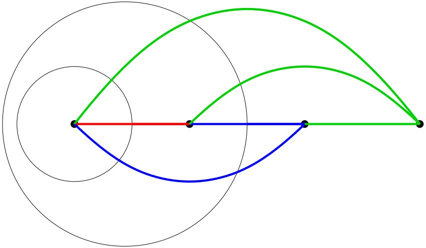
Proof of Theorem 24.65(2)



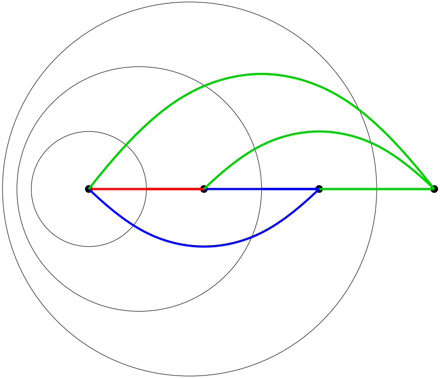
Proof of Theorem 24.65(2)



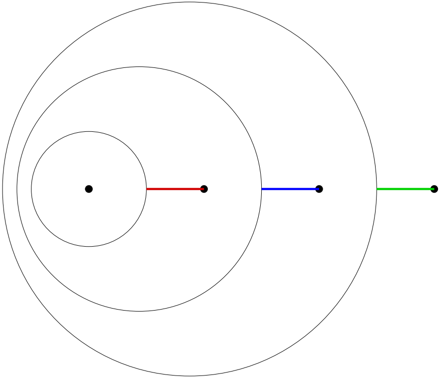
Proof of Theorem 24.65(2)



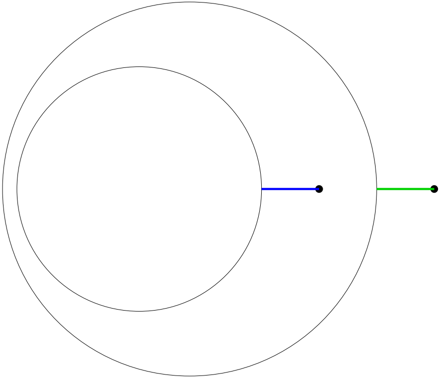
Proof of Theorem 24.65(2)



Proof of Theorem 24.65(2)



Proof of Theorem 24.65(2)



Spectrum $\{\omega\}$

Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.

Spectrum $\{\omega\}$

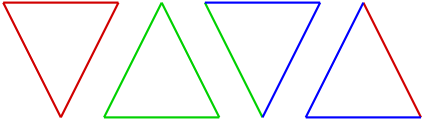
Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$. (9/2007)

Spectrum $\{\omega\}$

Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$. (9/2007)
Comer'86 proved ω is in the spectrum. The proof that the spectrum has no finite numbers is similar to Theorem 24.65, but much more complicated.

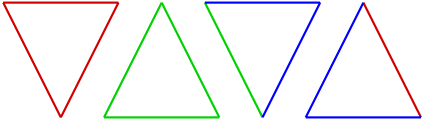
Spectrum $\{\omega\}$

Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$. (9/2007)
Comer'86 proved ω is in the spectrum. The proof that the spectrum has no finite numbers is similar to Theorem 24.65, but much more complicated.
If a =red, b =blue, c =green, then the forbidden triangles are

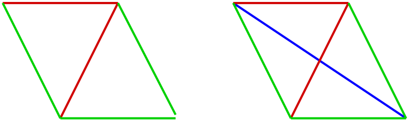


Spectrum $\{\omega\}$

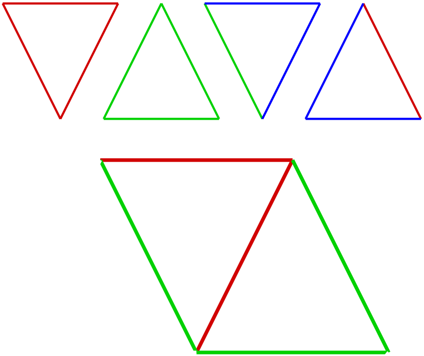
Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$. (9/2007)
Comer'86 proved ω is in the spectrum. The proof that the spectrum has no finite numbers is similar to Theorem 24.65, but much more complicated.
If a =red, b =blue, c =green, then the forbidden triangles are



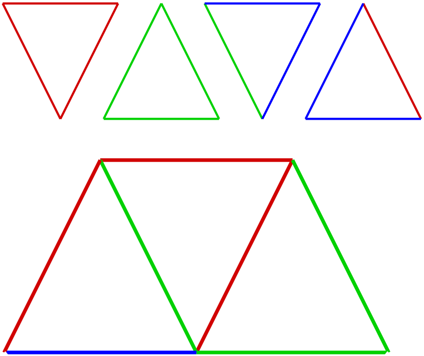
Lemma: If the left side occurs, then the missing edge must be blue, as shown.



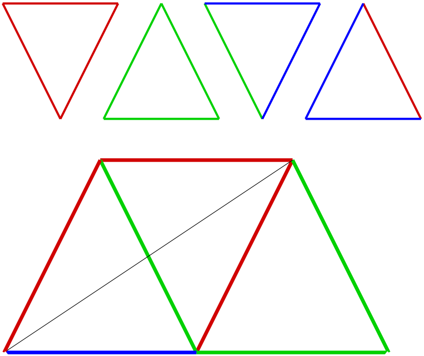
Proof of Lemma



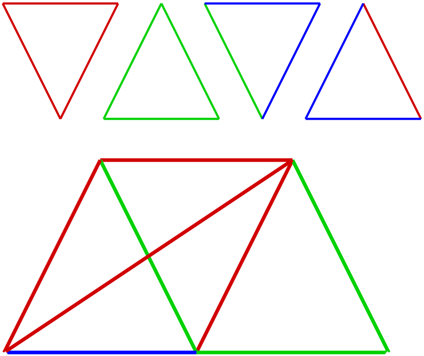
Proof of Lemma



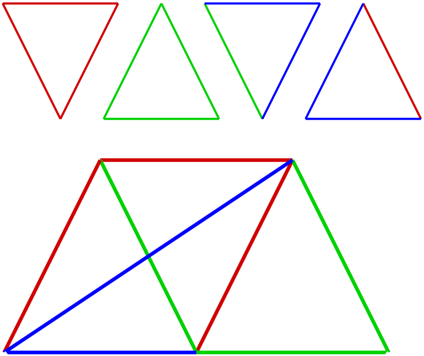
Proof of Lemma



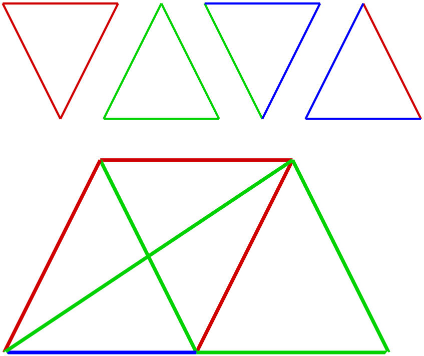
Proof of Lemma



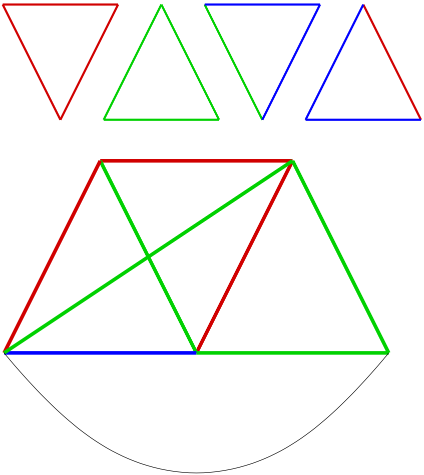
Proof of Lemma



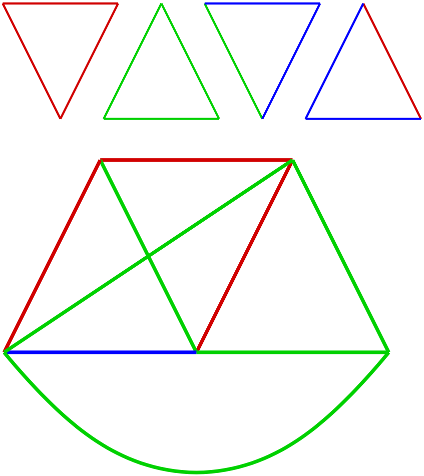
Proof of Lemma



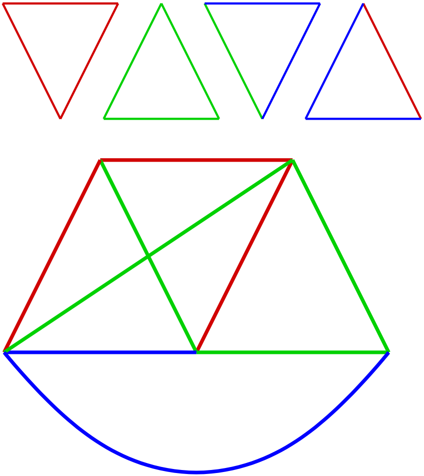
Proof of Lemma



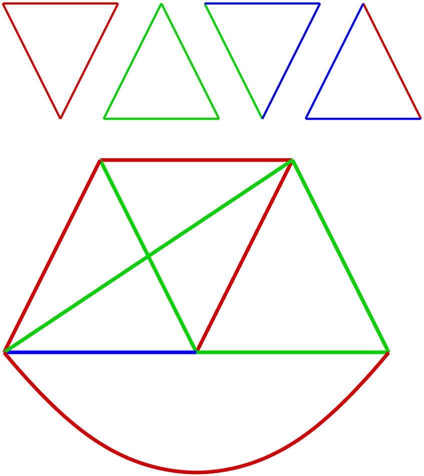
Proof of Lemma



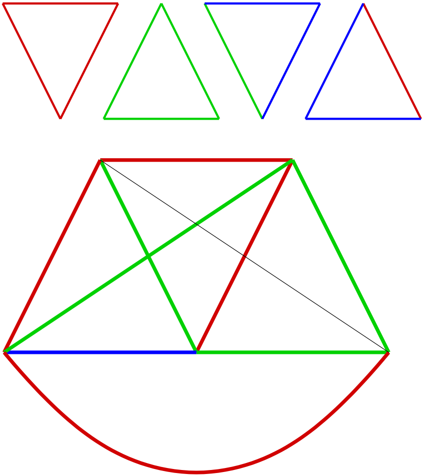
Proof of Lemma



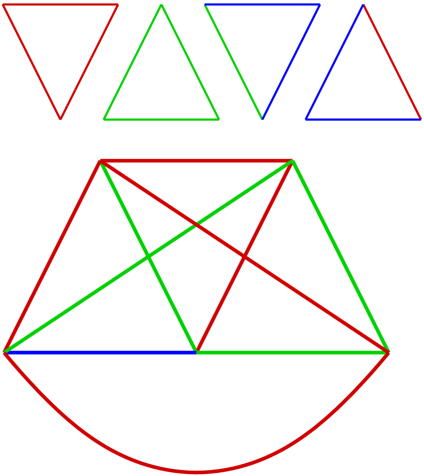
Proof of Lemma



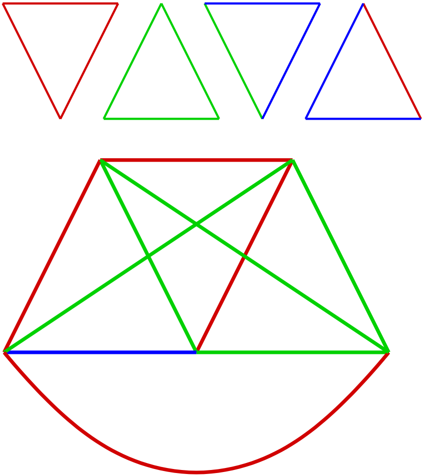
Proof of Lemma



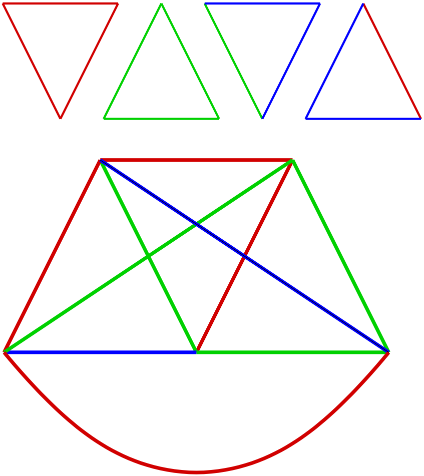
Proof of Lemma



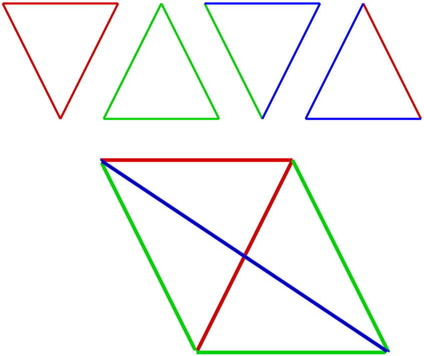
Proof of Lemma



Proof of Lemma



Proof of Lemma



Three related problems

Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$.
Problem 30.65: $Sp(a, b, c; bbb, acc, bcc, cbb) = \{\omega\}$?
Problem 52.65: $Sp(a, b, c; ccc, abb, cbb) = \{\omega\}$?
Problem 56.65: $Sp(a, b, c; aaa, ccc, cbb) = \{\omega\}$?

Three related problems

Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$.
Problem 30.65: $Sp(a, b, c; bbb, acc, bcc, cbb) = \{\omega\}$?
Problem 52.65: $Sp(a, b, c; ccc, abb, cbb) = \{\omega\}$?
Problem 56.65: $Sp(a, b, c; aaa, ccc, cbb) = \{\omega\}$?
Problem 30.65 is really interesting. The answer is probably "yes".

The Flexible Color Conjecture

Theorem: If some color in C is **flexible** (does not occur in any forbidden triangle in F), then $\omega \in Sp(C, F)$.

The Flexible Color Conjecture

Theorem: If some color in C is **flexible** (does not occur in any forbidden triangle in F), then $\omega \in Sp(C, F)$.
For example, $\omega \in Sp(a, b, c; bbb, ccc, bcc, cbb)$ since a is flexible, but

The Flexible Color Conjecture

Theorem: If some color in C is **flexible** (does not occur in any forbidden triangle in F), then $\omega \in Sp(C, F)$.
For example, $\omega \in Sp(a, b, c; bbb, ccc, bcc, cbb)$ since a is flexible, but
Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.

The Flexible Color Conjecture

Theorem: If some color in C is **flexible** (does not occur in any forbidden triangle in F), then $\omega \in Sp(C, F)$.
For example, $\omega \in Sp(a, b, c; bbb, ccc, bcc, cbb)$ since a is flexible, but
Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.
Theorem 32.65 is a special case of

The Flexible Color Conjecture

Theorem: If some color in C is **flexible** (does not occur in any forbidden triangle in F), then $\omega \in Sp(C, F)$.
For example, $\omega \in Sp(a, b, c; bbb, ccc, bcc, cbb)$ since a is flexible, but
Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.
Theorem 32.65 is a special case of
FCC (the Flexible Color Conjecture): If C has a flexible color, then $Sp(C, F)$ contains a finite number.

The Flexible Color Conjecture

Theorem: If some color in C is **flexible** (does not occur in any forbidden triangle in F), then $\omega \in Sp(C, F)$.
For example, $\omega \in Sp(a, b, c; bbb, ccc, bcc, cbb)$ since a is flexible, but
Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.
Theorem 32.65 is a special case of
FCC (the Flexible Color Conjecture): If C has a flexible color, then $Sp(C, F)$ contains a finite number.
Theorem 32.65 is also a special case of
Theorem (Alm, M, Manske 2008): FCC holds if F is the set of *all* edge-colored triangles in which the flexible color does not occur.

Three Problems

Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.

Three Related Problems: Do any of these spectra contain a finite number?

33.65: $Sp(a, b, c; ccc, bcc, cbb)$
34.65: $Sp(a, b, c; bcc, cbb)$
59.65: $Sp(a, b, c; bbb, cbb)$

Three Problems

Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.

Three Related Problems: Do any of these spectra contain a finite number?

33.65: $Sp(a, b, c; ccc, bcc, cbb)$

34.65: $Sp(a, b, c; bcc, cbb)$

59.65: $Sp(a, b, c; bbb, cbb)$

Note that a is flexible, so ω is in all three spectra.

Three Problems

Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.

Three Related Problems: Do any of these spectra contain a finite number?

33.65: $Sp(a, b, c; ccc, bcc, cbb)$

34.65: $Sp(a, b, c; bcc, cbb)$

59.65: $Sp(a, b, c; bbb, cbb)$

Note that a is flexible, so ω is in all three spectra. Any "no" would disprove the FCC.

Three Problems

Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.

Three Related Problems: Do any of these spectra contain a finite number?

33.65: $Sp(a, b, c; ccc, bcc, cbb)$

34.65: $Sp(a, b, c; bcc, cbb)$

59.65: $Sp(a, b, c; bbb, cbb)$

Note that a is flexible, so ω is in all three spectra. Any "no" would disprove the FCC. Perhaps "yes" can be proved by elaborating the proof of Theorem 32.65.

What's known and isn't

Let $C = \{a, b, c\}$ and let F be a subset of $\{bbb, ccc, bcc, cbb\}$.

name	F				spectrum has
32.65	bbb	ccc	bcc	cbb	some $n < \omega$
33.65	–	ccc	bcc	cbb	?
34.65	–	–	bcc	cbb	?
57.65	–	ccc	–	cbb	$\binom{8}{3} = 56$
59.65	bbb	–	–	cbb	?
61.65	–	–	–	cbb	$\binom{9}{3} = 84$
64.65	–	ccc	–	–	25
65.65	–	–	–	–	19

What's known and isn't

Let $C = \{a, b, c\}$ and let F be a subset of $\{bbb, ccc, bcc, cbb\}$.

name	F				spectrum has
32.65	bbb	ccc	bcc	cbb	some $n < \omega$
33.65	–	ccc	bcc	cbb	?
34.65	–	–	bcc	cbb	?
57.65	–	ccc	–	cbb	$\binom{8}{3} = 56$
59.65	bbb	–	–	cbb	?
61.65	–	–	–	cbb	$\binom{9}{3} = 84$
64.65	–	ccc	–	–	25
65.65	–	–	–	–	19

The last line, which says $19 \in Sp(a, b, c;)$, is a special case of
Theorem (Jipsen, M, Tuza): The FCC holds if $F = \emptyset$.