Edge-coloring problems

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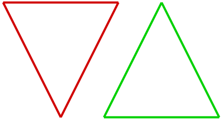
A classical edge-coloring problem

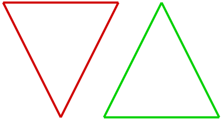
Given: a finite set C of colors and some forbidden triangles (copies of K_3) whose edges are colored with colors in C.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears?
Example: The colors are red and green and the forbidden triangles are the two monochromatic ones:



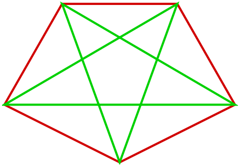
Theorem: The edges of K_n can be 2-colored so that no monochromatic triangle appears iff $n \in \{1, 2, 3, 4, 5\}$.

Solution to a classic coloring problem

Proof: The colors are red and green and the forbidden triangles are



The edges of K_n can be 2-colored so that no monochromatic triangle appears if $n \in \{1, 2, 3, 4, 5\}$. Here's how:



The subgraphs give 2-colorings for K_1, K_2, K_3, K_4.

The Party Theorem: If $n \geq 6$ then every 2-coloring of the edges K_n contains a monochromatic triangle.

A variation on the classic problem

Given: a finite set of colors and some forbidden edge-colored triangles.
Problem: For which n is it possible to color the edges of K_n so that no forbidden triangle appears and *whatever is not forbidden is mandatory*? (a **proper** coloring)

"Whatever is not forbidden is mandatory" means
every vertex must be part of an edge of each color, and
if a colored edge could be part of a non-forbidden triangle, then it is.

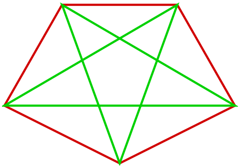
Definition: If F is a finite set of triangles whose edges have been colored with colors in a finite set C, then the **spectrum** $Sp(C,F)$ is the set of countable cardinals n such that the edges of K_n can be properly colored using colors in C (no triangle in F occurs, and everything mandatory *does* occur). $Sp(C,F) \subseteq \{1,2,3,\dots,\omega\}$.

The previous example, revisited under the new rules

There are two colors, so $C = \{a, b\}$, and the forbidden triangles are the two monochromatic ones, which we'll call *aaa* and *bbb*, so $F = \{aaa, bbb\}$.

Theorem 5.7: $Sp(C, F) = Sp(a, b; aaa, bbb) = \{5\}$.

Proof: The edge-colored K_5 below avoids monochromatic triangles and contains everything mandatory. No subgraph of this graph has everything it needs, and neither does any other 2-coloring of the edges of K_n where $n \leq 4$. (HW: Check this!)



Some general considerations

- There are $\binom{|C|+2}{3}$ triangles with C-colored edges.

For example, 4 triangles if $|C| = 2$, and 10 triangles if $|C| = 3$.

- For *many* sets F of colored triangles, $Sp(C, F) = \emptyset$.

For example, $Sp(a, b; aaa, bbb, baa, abb) = \emptyset = Sp(a, b; baa, abb)$.

- A necessary condition for $Sp(C, F) \neq \emptyset$ is that if $a, b, c, e, d \in C$ and $abc, ced \notin F$ then there is some $f \in C$ such that $aef, fbd \notin F$.

Only examples satisfying this condition are considered below.

- By the Compactness Theorem, if $Sp(C, F)$ contains arbitrarily large integers, then $\omega \in Sp(C, F)$.

More results on two colors

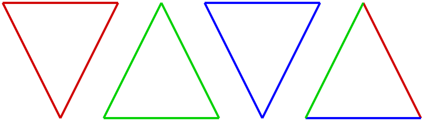
Theorem 1.7: $Sp(a, b; aaa, bbb, baa) = \{4\}$. Proof: consider $\mathbb{Z}_2^2$.
Theorem 2.7: $Sp(a, b; bbb, baa) = \{6, \cdots, \omega\}$.
Theorem 3.7: $Sp(a, b; aaa, baa) = \{6, 8, 10, \cdots, \omega\}$.
Theorem 4.7: $Sp(a, b; baa) = \{9, \cdots, \omega\}$.
Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 6.7: $Sp(a, b; bbb) = \{8, \cdots, \omega\}$.
Theorem 7.7: $Sp(a, b;) = \{9, \cdots, \omega\}$. (9: U. Wostner, ∼1976)

More on forbidding monochromatic triangles

Theorem 5.7: $Sp(a, b; aaa, bbb) = \{5\}$.
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Proof: On $\{0, \cdots, 12\} = \mathbb{Z}_{13}$, color ij with a if $i - j \in \{1, 5, 8, 12\}$, b if $i - j \in \{2, 3, 10, 11\}$.
17 or more points can't be properly colored by Greenwood-Gleason'55.
The two good 3-colorings of K_{16} of Greenwood-Gleason'55 and Kalbfleisch-Stanton'68 are proper.
Kramer proved 14 and 15 are impossible.
Theorem (Ramsey): $\omega \notin Sp(C, \{ccc : c \in C\})$.
Conjecture: $Sp(C, \{ccc : c \in C\}) \neq \emptyset$.
True for $|C| = 2$ by Theorem 5.7, for $|C| = 3$ by Theorem 62.65, for $|C| = 4, 5$ by Comer'83.
Erdös, Szemeredi, and Trotter proved it for all sufficiently large finite C,
but they were wrong. Try it for 6 colors!

More results on three colors

Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
A contradiction can be found by looking at only 5 points. If $a =$red, $b =$blue, $c =$green, then the forbidden triangles are

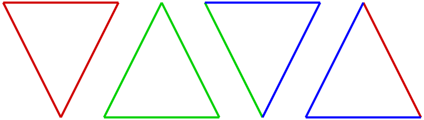


More results on three colors

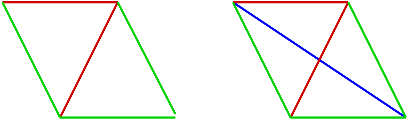
Theorem 62.65: $Sp(a, b, c; aaa, bbb, ccc) = \{13, 16\}$.
Theorem 21.65: $Sp(a, b, c; aaa, bbb, ccc, abc) = \emptyset$.
Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$. (∼1974)
The proof has two parts: (1) K_ω can be properly colored.
(2) If K_n is properly colored, then $n \geq \omega$.
This is a "surprising" result. For example, in 1994 A.Simon conjectured (only briefly, at a meeting) that $\{\omega\}$ cannot be a spectrum.

Spectrum $\{\omega\}$

Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$. (9/2007)
Comer'86 proved ω is in the spectrum. The proof that the spectrum has no finite numbers is similar to Theorem 24.65, but much more complicated.
If a =red, b =blue, c =green, then the forbidden triangles are



Lemma: If the left side occurs, then the missing edge must be blue, as shown.



Three related problems

Theorem 24.65: $Sp(a, b, c; abc) = \{\omega\}$.
Theorem 51.65: $Sp(a, b, c; aaa, ccc, abb, cbb) = \{\omega\}$.
Problem 30.65: $Sp(a, b, c; bbb, acc, bcc, cbb) = \{\omega\}$?
Problem 52.65: $Sp(a, b, c; ccc, abb, cbb) = \{\omega\}$?
Problem 56.65: $Sp(a, b, c; aaa, ccc, cbb) = \{\omega\}$?
Problem 30.65 is really interesting. The answer is probably "yes".

The Flexible Color Conjecture

Theorem: If some color in C is **flexible** (does not occur in any forbidden triangle in F), then $\omega \in Sp(C, F)$.
For example, $\omega \in Sp(a, b, c; bbb, ccc, bcc, cbb)$ since a is flexible, but
Theorem 32.65: $Sp(a, b, c; bbb, ccc, bcc, cbb)$ contains a finite number.
Theorem 32.65 is a special case of
FCC (the Flexible Color Conjecture): If C has a flexible color, then $Sp(C, F)$ contains a finite number.
Theorem 32.65 is also a special case of
Theorem (Alm, M, Manske 2008): FCC holds if F is the set of *all* edge-colored triangles in which the flexible color does not occur.

Three Problems

Theorem 32.65: $Sp(a,b,c;bbb,ccc,bcc,cbb)$ contains a finite number.
Three Related Problems: Do any of these spectra contain a finite number?
33.65: $Sp(a,b,c;ccc,bcc,cbb)$
34.65: $Sp(a,b,c;bcc,cbb)$
59.65: $Sp(a,b,c;bbb,cbb)$
Note that a is flexible, so ω is in all three spectra.
Any “no” would disprove the FCC.
Perhaps “yes” can be proved by elaborating the proof of Theorem 32.65.

What's known and isn't

Let $C = \{a, b, c\}$ and let F be a subset of $\{bbb, ccc, bcc, cbb\}$.

name	F				spectrum has
32.65	bbb	ccc	bcc	cbb	some $n < \omega$
33.65	–	ccc	bcc	cbb	?
34.65	–	–	bcc	cbb	?
57.65	–	ccc	–	cbb	$\binom{8}{3} = 56$
59.65	bbb	–	–	cbb	?
61.65	–	–	–	cbb	$\binom{9}{3} = 84$
64.65	–	ccc	–	–	25
65.65	–	–	–	–	19

The last line, which says $19 \in Sp(a, b, c;)$, is a special case of **Theorem** (Jipsen, M, Tuza): The FCC holds if $F = \emptyset$.