FINITE, INTEGRAL, AND FINITE-DIMENSIONAL RELATION ALGEBRAS: A BRIEF HISTORY.

ROGER D. MADDUX

ABSTRACT. Relation were invented by Tarski and his collaborators in the middle of the twentieth century. The concept of integrality arose naturally early in the history of the subject, and so did various constructions of finite integral relation algebras. Later the concept of finite-dimensionality was introduced for classifying nonrepresentable relation algebras. This concept is closely connected to the number of variables used in proofs in first-order logic. I recount some results on these topics in chronological order.

1. THE CALCULUS OF RELATIONS AND FINITE-VARIABLE LOGIC: 1940–1980.

The **relative product** $R|S$ of two binary relations R and S is defined by

$$R|S := \{\langle x, y\rangle : \exists_x (xRz \wedge zSy)\}.$$

As a symbol for relative multiplication, the vertical stroke | was introduced by A. N. Whitehead and B. Russell in Principia Mathematica [33, 34, 35]. It is easy to prove that relative multiplication obeys the associative law

$$R|(S|T) = (R|S)|T. \tag{1}$$

For a proof of the inclusion from left to right, suppose that $\langle a, b\rangle \in R|(S|T)$. Then there must be some c such that $\langle a, c\rangle \in R$ and $\langle c, b\rangle \in S|T$. By the latter statement, there must be some d such that $\langle c, d\rangle \in S$ and $\langle d, b\rangle \in T$. From $\langle a, c\rangle \in R$ and $\langle c, d\rangle \in S$ we conclude that $\langle a, d\rangle \in R|S$, which, combined with $\langle d, b\rangle \in T$, gives $\langle a, b\rangle \in (R|S)|T$. Note that four objects, namely a, b c, d, are used in the proof. This much was apparent to A. De Morgan[21, 22, 24, 23], C. S. Peirce [26, 27, 28], and E. Schröder [30]. A. Tarski [31, 32] discovered that the reference to four objects is *required* to prove (1); it can't be proved with reference to only three objects.

To see how to prove this formally, start by restating (1) as a sentence in a first-order language that has only binary relations symbols. The equation $R = S$ between two relations is expressed by the sentence

$$\forall_x \, \forall_y \, (xRy \Leftrightarrow xSy).$$

Applying this to (1) gives

$$\forall_x \, \forall_y \, \Big(x(R|S)|Ty \Leftrightarrow xR|(S|T)y\Big).$$

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Expand the formulas $x(R|S)|Ty$ and $xR|(S|T)y$ according to the definition of relative multiplication, using a third variable, z:

$$\forall_x \forall_y \Big(\underbrace{x(R|S)|Ty}_{\exists_z (xR|Sz \wedge zTy)} \Leftrightarrow \underbrace{xR|(S|T)y}_{\exists_z (xRz \wedge zS|Ty)} \Big).$$

After substitution, the next step is to expand the formulas $xR|Sz$ and $zS|Ty$. This time, the variable y does not occur in $xR|Sz$, so y can be used to express $xR|Sz$ as $\exists_y (xRy \wedge ySz)$, and x does not occur in $zS|Ty$, so $zS|Ty$ is equivalent to $\exists_x (zSx \wedge xTy)$:

$$\forall_x \forall_y \Big(\exists_z (\underbrace{xR|Sz}_{\exists_y (xRy \wedge ySz)} \wedge zTy) \Leftrightarrow \exists_z (xRz \wedge \underbrace{zS|Ty}_{\exists_x (zSx \wedge xTy)})\Big).$$

Substitute and conclude that (1) is equivalent to

$$\forall_x \forall_y \Big(\exists_z (\exists_y (xRy \wedge ySz) \wedge zTy) \Leftrightarrow \exists_z (xRz \wedge \exists_y (zSx \wedge xTy))\Big).$$

The only variables needed for this translation are x, y, and z.

The converse R^{-1} of the binary relation R is defined by

$$R^{-1} := \{\langle x, y\rangle : yRx\}.$$

The following law is similar to a familiar equation from group theory.

$$(R|S)^{-1} = S^{-1}|R^{-1}. \tag{2}$$

Translate (2) into first-order logic:

$$\forall_x \forall_y \Big(\underbrace{x(R|S)^{-1}y}_{yR|Sx} \Leftrightarrow \underbrace{xS^{-1}|R^{-1}y}_{\exists_z (xS^{-1}z \wedge zR^{-1}y)} \Big),$$

$$\forall_x \forall_y \Big(\underbrace{yR|Sx}_{\exists_z (yRz \wedge zSx)} \Leftrightarrow \exists_z (\underbrace{xS^{-1}z}_{zSx} \wedge \underbrace{zR^{-1}y}_{yRz}) \Big),$$

$$\forall_x \forall_y \Big(\exists_z (yRz \wedge zSx) \Leftrightarrow \exists_z (zSx \wedge yRz) \Big).$$

This computation can serve as a proof of (2), since the last line is clearly true and can be proved on the basis of fairly simple principles of logic, such as *modus ponens* and the substitutivity of equivalent formulas:

$$\frac{\varphi \qquad \varphi \Rightarrow \psi}{\psi} \qquad\qquad \frac{\varphi \Leftrightarrow \psi}{\exists_x \varphi \Leftrightarrow \exists_x \psi} \qquad\qquad \frac{\varphi \Leftrightarrow \psi}{\forall_x \varphi \Leftrightarrow \forall_x \psi}$$

Other relations that can be made from R and S are the Boolean ones, namely, the union $R \cup S$, complement $\overline{R}$ (relative to some universal relation called 1), and intersection $R \cap S$. Let Id be the identity relation on field of the universal relation 1. Let $\{R, S, \dots\}$ be a countable set of variables ranging over binary relations. Build terms denoting binary relations using the variables R, S, ..., the names of the specific relations 1 and Id, and the operation symbols for $\cap$, $\cup$, $^-$, $|$, and $^{-1}$. Pairs of such terms are called **equations in the calculus of relations**. Every such equation E can be translated, in a way illustrated by the examples above, into a sentence σ_E in the first-order language $\mathcal{L}_3$, whose quantifiers and connectives are $\exists$, $\forall$, $\wedge$, $\vee$, $\neg$, $\Leftrightarrow$, and $\Rightarrow$, whose binary relation symbols are the variables $R, S, \dots$, and whose only individual variables are x, y, and z. The equation E is equivalent to its translation σ_E, so every equation is equivalent to a sentence in $\mathcal{L}_3$. Tarski proved the converse, that every sentence in $\mathcal{L}_3$ is equivalent to an equation. For a much stronger version of the following theorem that involves a recursive translation function, see [32].

Theorem 1. (Tarski [32]) *For every τ in $\mathcal{L}_3$ there is some equation E such that $\sigma_E \equiv_3 \tau$.*

Here $\equiv_3$ denotes semantic equivalence in $\mathcal{L}_3$, but Tarski showed that the theorem is still true when $\equiv_3$ denotes *provable* equivalence. To properly formulate this notion, use any standard textbook axiomatization of first-order logic (restricted to 3 variables and binary relation symbols), and make up for any deficiencies by adding as axioms all sentences that assert the equivalence of a formula and one obtained from it by the process known as "respelling bound variables". (In fact, even more care is required. The equations of the calculus of relations and the sentences of $\mathcal{L}_3$ must be taken to be part of a still larger language called $\mathcal{L}_3^+$, and sentences expressing the translation principles illustrated above must be added as axioms. See [32] for details.)

We saw above that (2) translates into a sentence that is provable in $\mathcal{L}_3$: use commutativity of conjunction and substitutivity of biconditionals. Other equations whose translations are provable in $\mathcal{L}_3$ are:

$$R \cup (S \cup T) = (R \cup S) \cup T, \tag{3}$$

$$R \cup S = S \cup R, \tag{4}$$

$$\overline{\overline{R}} = R, \tag{5}$$

$$1 = R \cup \overline{R}, \tag{6}$$

$$R \cap S = \overline{\overline{R} \cup \overline{S}}, \tag{7}$$

$$R = (R \cap S) \cup \left(R \cap \overline{S}\right), \tag{8}$$

$$R|(S \cup T) = R|S \cup R|T, \tag{9}$$

$$\mathsf{Id}|R = R, \tag{10}$$

$$R^{-1-1} = R, \tag{11}$$

$$\overline{S} = R|\overline{R|S} \cup \overline{S}. \tag{12}$$

Equations (1)–(12) happen to form an equational axiomatization of the class RA of relation algebras. Consequently, an equation is equationally derivable from (1)–(12) just in case it is valid in every relation algebra. Except for the associative law, these equations and their consequences are 3-provable.

Theorem 2. (Tarski) *If E is equationally derivable from* (2)–(12)*, then σ_E is provable in $\mathcal{L}_3$.*

The exceptional property of the associative law leads naturally to the question of its algebraic independence from the remaining axioms. Tarski's colleague J. J. C. McKinsey at Stanford found an algebra that satisfies all the axioms for relation algebras *except* (1).

Theorem 3. (McKinsey) (1) *is not equationally derivable from* (2)–(12).

Let NA be the class of algebras whose equational axiom set is (2)–(12). McKinsey's theorem says that $\mathsf{NA} \subset \mathsf{RA}$.

It should be clear from the informal proof of (1) given earlier, that (1) is 4-provable, that is, $\sigma_{(1)}$ is provable in $\mathcal{L}_4$, the first-order language of binary relation symbols that has four individual variables x, y, z, and w, instead of just three. Tarski used McKinsey's algebra to show that three variables are not enough to prove (1).

Theorem 4. (Tarski) *$\sigma_{(1)}$ is provable in $\mathcal{L}_4$, but not in $\mathcal{L}_3$.*

The material presented so far was known to Tarski around 1950, but many of the details were first published in 1987 [32]. In fact, Tarski never published his proof of the non-3-provability of (1), but a proof that uses cylindric algebras was published by L. Henkin.

Theorem 5. (Henkin [3]) *$\sigma_{(1)}$ is not provable in $\mathcal{L}_3$.*

In view of the results above, Tarski asked whether there an equation E that is 3-provable (σ_E is provable in $\mathcal{L}_3$), but not equationally derivable from (2)–(12) (not valid in every NA). While working on my dissertation under Tarski, I discovered that such equations do exist. One example is the semi-associative law:

$$(R|1)|1 = R|1. \tag{13}$$

The equation $1 = 1|1$ is derivable from (2)–(12), hence is valid in every NA, so the semi-associative law is equivalent in NA to an instance of the associative law, namely,

$$(R|1)|1 = R|(1|1).$$

Let SA be the class of algebras whose equational axiom set is (2)–(13). (My first name for this class was "TA", for "Tarski algebras". Later it was "SRA", but Tarski warned me against using this letter combination because of its meaning in Polish.) I proved that (13) is not derivable from (2)–(12) by constructing algebras that are in SA and yet violate (13), one from a loop with the inverse property and another from a 9-element Steiner triple system [11, p. 52–55]. McKinsey's algebra does not satisfy (13), which accounts for the second inclusion in the following theorem.

Theorem 6. [11, p. 61] $\mathsf{RA} \subset \mathsf{SA} \subset \mathsf{NA}$.

Tarski's question about 3-provability, stated in terms of NA, is whether there are 3-provable equations that fail in some NA. One might reasonably repeat Tarski's question for SA: are there any 3-provable equations that fail in some SA? This time I found that the answer is "no". There are many less special cases of (1) than (13) that are 3-provable, such as

$$(R|S)|1 = R|(S|1),$$
$$(R|1)|S = R|(1|S),$$
$$(1|R)|S = 1|(R|S),$$

etc., but these are all valid in SA. In fact, the axiom set for SA is complete for 3-provability.

Theorem 7. [11, Th. 11(30)] *For any equation E, the following are equivalent:*

- *E is equationally derivable from* (2)–(13)*,*
- *E is valid in every* SA*,*
- *E is 3-provable.*

Furthermore, RA is complete for 4-provability, providing a characterization of Tarski's axiom set.

Theorem 8. [11, Th. 11(31)] *For any equation E, the following are equivalent:*

- *E is equationally derivable from* (1)–(12)*,*
- *E is valid in every* RA*,*
- *E is 4-provable.*

Until this theorem, Tarski's axiom set for RA was distinguished only by being simple, natural, and capable of proving "all of the hundreds of theorems to be found in Schröder's *Algebra und Logik der Relative*" [31, pp. 87–88]. The set of all subrelations of an equivalence relation $\mathcal{E}$ is closed under $\cap$, $\cup$, $^-$ (relative to $\mathcal{E}$), $|$, and $^{-1}$, and contains the identity relation on the field of $\mathcal{E}$, so it forms the universe of an algebra that can easily be seen to satisfy all the axioms for relation algebras. A relation algebra is said to be **representable** if it is isomorphic to a subalgebra of the relation algebra of all subrelations of some equivalence relation. RRA is the **class of representable relation algebras**. R. Lyndon [8] proved that nonrepresentable relation algebras exist (theorem 17 below). This shows that Tarski's axiom set is not complete. Furthermore, using results of Lyndon [10] (theorem 19 below), J. D. Monk [19] proved that RRA is not finitely axiomatizable. It follows that Tarski's axiom set is not only incomplete, but it cannot be completed by adding any finite set of equations. Why is Tarski's axiom set special? Or is it? I offer theorem 8 as an answer.

The proofs of theorems 7 and 8 require algebraic replacements for various notions that arise in the semantics of $\mathcal{L}_3$ (and $\mathcal{L}_4$). Suppose $\mathcal{M}$ is a model for $\mathcal{L}_3$. Since $\mathcal{L}_3$ has only three variables x, y, and z, the definition of satisfaction only needs sequences of 3 elements in the domain M of $\mathcal{M}$, such as $\langle a_0, a_1, a_2\rangle$, with the convention that the first element a_0 is assigned to the variable x, the second to y, and the third to z. Part of the definition of satisfaction reads,

$$\mathcal{M} \models xRy\,[\langle a_0, a_1, a_2\rangle] \text{ iff } \langle a_0, a_1\rangle \in R^{\mathcal{M}}.$$

The natural analogue of $R^{\mathcal{M}}$ is an element in a relation algebra, in imitation of Henkin [3]. The problem is to determinine algebraic counterparts for the sequences $\langle a_0, a_1, a_2\rangle$ and $\langle a_0, a_1\rangle$. Since $\{\langle a_0, a_1\rangle\}$ is an atom in the algebra of all binary relations, atoms take the place of ordered pairs. The condition $\langle a_0, a_1\rangle \in R^{\mathcal{M}}$ is equivalent to the inclusion $\{\langle a_0, a_1\rangle\} \subseteq R^{\mathcal{M}}$, which becomes an algebraic inequality in some relation algebra. Such inclusions are the only information needed from the sequence $\langle a_0, a_1, a_2\rangle$. Therefore, a list containing one atom for each pair of indices can replace the sequence $\langle a_0, a_1, a_2\rangle$. Such a list is conveniently displayed as a matrix, with the atom associated with indices i and j (for $0 \le i, j \le 2$) in row $i+1$ and column $j+1$. Now proceed more formally. Assume $\mathfrak{A} \in \mathsf{NA}$, $\mathfrak{A}$ is atomic, and $3 \le n \le \omega$. Let $B_n\mathfrak{A}$ be the set of n-by-n matrices m of atoms of $\mathfrak{A}$ that satisfy the following conditions for all indices i and j:

$$m_{ii} \le 1', \quad (m_{ij})^{\smile} = m_{ji}, \quad m_{ij} \le m_{ik}\,;m_{kj}.$$

Elements of $B_n\mathfrak{A}$ are called **basic matrices**, and M is said to be an **n-dimensional relational basis for** $\mathfrak{A}$ if $M \subseteq B_n\mathfrak{A}$, every atom of $\mathfrak{A}$ appears in some $m \in M$, and

(ext) if $m \in M$, $i, j < n$, $a, b \in At\mathfrak{A}$, $m_{ij} \le a;b$, and $i, j \ne k < n$, then there is some $m' \in M$ such that $m'_{ik} = a$, $m'_{kj} = b$, and $m_{ij} = m'_{ij}$ whenever $k \ne i, j$.

Each basis M has a corresponding "complex algebra", which can be described as a vector space over the 2-element field, with dimension equal to the cardinality of M, generated by the set M of vectors. Here are two crucial theorems about relational bases.

Theorem 9. [12, Th. 4] *If $\mathfrak{A}$ is an atomic SA then $B_3\mathfrak{A}$ is a 3-dimensional relational basis for $\mathfrak{A}$.*

Theorem 10. [12, Th. 5] *If $\mathfrak{A}$ is an atomic RA then $B_4\mathfrak{A}$ is a 4-dimensional relational basis for $\mathfrak{A}$.*

Next is a brief sketch of the proof of theorem 8. The equivalence of the first two conditions is just the definition of RA. Assume the equation E is equationally derivable from the axioms (1)–(12). Each of these axioms is 4-provable, and so are their equational consequences, so E is 4-provable. For the converse, suppose that E is not derivable from (1)–(12). Then E must fail in some $\mathfrak{A} \in \mathsf{RA}$. Every relation algebra has a perfect extension $\mathfrak{A}^+ \in \mathsf{RA}$ [7], which is, among other things, a complete atomic relation algebra that contains $\mathfrak{A}$ as a subalgebra. Since E fails in $\mathfrak{A}$, it must also fail in its superalgebra $\mathfrak{A}^+$. By theorem 10, $B_4\mathfrak{A}^+$ is a 4-dimensional relational basis for $\mathfrak{A}^+$. This makes it possible to construct an alternative algebraic model $\mathcal{M}$ for $\mathcal{L}_4$. $\mathcal{M}$ must assign each binary relation symbol R in $\mathcal{L}_4$ to an element $R^{\mathcal{M}}$ of $\mathfrak{A}^+$. If R is a variable in E, then let $R^{\mathcal{M}}$ be the element of $\mathfrak{A}^+$ to which R is assigned in the failure of E in $\mathfrak{A}^+$, and otherwise let $R^{\mathcal{M}}$ be *any* element of $\mathfrak{A}^+$. Begin the definition of satisfaction as follows. For any $m \in B_4\mathfrak{A}^+$ and any relation symbol R of $\mathcal{L}_4$,

$$\mathcal{M} \models xRy\,[m] \text{ iff } m_{01} \le R^{\mathcal{M}},$$

and so on for other combinations of variables besides x and y. Add appropriate clauses for the connectives and the quantifiers. It can then be shown that $\mathcal{M}$ is an alternative algebraic model in which *all* 4-provable sentences are valid, but E is *not* valid in $\mathcal{M}$ (neither equationally nor under the alternative semantics), so E is not 4-provable. The same proof works for theorem 7; just change 4 to 3 and add "semi-associative" where needed.

2. Finite-dimensional algebras: 1980–2000.

The results in the previous section involving the numbers 3 and 4 cannot be extended to 5, 6, and so on. The concept of dimension helps to explain why. Let $3 \leq n \leq \omega$. Say that $\mathfrak{A}$ is a **relation algebra of dimension** n if $\mathfrak{A}$ is a subalgebra of an atomic NA that has an n-dimensional relational basis. Let RA_n be the **class of all relation algebras of dimension** n. Let the **dimension** of a semi-associative relation algebra $\mathfrak{A} \in \mathsf{SA}$ be the largest $n < \omega$ such that $\mathfrak{A} \in \mathsf{RA}_n \sim \mathsf{RA}_{n+1}$, and ω if there is no such n. The next theorem follows primarily from theorems 9 and 10, and shows that algebras in $\mathsf{SA} \sim \mathsf{RA}$ have dimension 3.

Theorem 11. [12, Th. 6(1)(2)] $\mathsf{SA} = \mathsf{RA}_3$, $\mathsf{RA} = \mathsf{RA}_4$.

Theorems 9 and 10 may suggest that the set of n-by-n basic matrices of an atomic algebra $\mathfrak{A}$ in RA_n is an n-dimensional relational basis, but it is easy to find, for example, a finite algebra $\mathfrak{A} \in \mathsf{RA}_5$ such that $B_5\mathfrak{A}$ is not a 5-dimensional relational basis for $\mathfrak{A}$. RA_n is an equational class (a variety) because it is closed under the formation of homomorphic images, direct products, and subalgebras:

Theorem 12. [12, Th. 9] $\mathsf{RA}_n = \mathbf{HSP}\,\mathsf{RA}_n$ *for* $3 \leq n \leq \omega$.

RA_n is a canonical variety because it is closed under the formation of perfect (also called canonical) extensions:

Theorem 13. [12, Th. 8] *If* $\mathfrak{A} \in \mathsf{RA}_n$ *then* $\mathfrak{A}^+$ *has an* n*-dimensional relational basis and* $\mathfrak{A}^+ \in \mathsf{RA}_n$.

The finite-dimensional varieties RA_n form a chain of varieties that converges on the variety of representable relation algebras:

Theorem 14. [12, Th. 3, 6(3), 10] $\mathsf{RA}_3 \supseteq \mathsf{RA}_4 \supseteq \mathsf{RA}_5 \supseteq \mathsf{RA}_6 \supseteq \cdots \supseteq \bigcap_{3 \leq n < \omega} \mathsf{RA}_n = \mathsf{RRA}$.

Thus a semi-associative relation algebra is finite-dimensional iff it is a nonrepresentable relation algebra. The dimension classifies nonrepresentable algebras. Algebras of with high finite dimension are "closer to representable" than algebras with low finite dimension. Is there a finite relation algebra of each finite dimension? Yes, but this question remained unanswered until the early 1990's. Indeed, all inclusions are strict:

Theorem 15. [15] $\mathsf{RA}_3 \supset \mathsf{RA}_4 \supset \mathsf{RA}_5 \supset \mathsf{RA}_6 \supset \cdots$

It seemed reasonable to conjecture that, in fact, all inclusions after the first are not even finitely axiomatizable [12, p.90], but several more years passed before this was finally proved by R. Hirsch and I. Hodkinson.

Theorem 16. (Hirsch-Hodkinson [4]) *If* $4 \leq n < \omega$ *then* RA_{n+1} *is not finitely axiomatizable relative to* RA_n.

3. Finite algebras: 1950–2000.

Tarski [31] asked whether the axioms for RA are complete, and whether all relation algebras are representable. Lyndon showed that the answer to both questions is "no".

Theorem 17. (Lyndon [8]) *There are nonrepresentable relation algebras with 52 and 56 atoms.*

Chin and Tarski [1] observed that equation (L) is valid in RRA but fails in Lyndon's algebras.

(L)
$$x_{02}\,;x_{21} \cdot x_{03}\,;x_{31} \cdot x_{04}\,;x_{41}$$
$$\leq x_{02}\,;\Big(x_{20}\,;x_{03} \cdot x_{21}\,;x_{13} \cdot (x_{20}\,;x_{04} \cdot x_{21}\,;x_{14})\,;(x_{40}\,;x_{03} \cdot x_{41}\,;x_{13})\Big)\,;x_{31}.$$

A relation algebra is **integral** iff $x\,;y = 0$ implies $x = 0$ or $y = 0$. An condition equivalent to integrality for relation algebras is that 1' is an atom [7]. This characterization is valid for semi-associative relation algebras [14, Th. 4], but not for nonassociative relation algebras [13, Th. 2]. McKinsey and Tarski saw how to construct representable relation algebras from groups [7]. Such algebras are integral. Lyndon's first nonrepresentable relation algebras were not integral, but B. Jónsson found a way to get an integral nonrepresentable relation algebra from a non-Desarguesian projective plane.

Theorem 18. (Jónsson [6]) *There is an integral nonrepresentable relation algebra with ω atoms.*

Jónsson proved that his algebra is not representable by observing that (J) is valid in RRA but fails in his algebra.

$$\text{(J)}\qquad \begin{aligned}&\text{if } x_{01} \le x_{02};x_{21} \cdot x_{03};x_{31}\\ &\text{and } x_{20};x_{03} \cdot x_{21};x_{13} \le x_{24};x_{43}\\ &\text{then } x_{01} \le (x_{02};x_{24} \cdot x_{03};x_{34});(x_{42};x_{20} \cdot x_{43};x_{31}).\end{aligned}$$

Lyndon modified Jónsson's construction, obtaining an integral relation algebra $\mathfrak{A}(\mathfrak{G})$ from each projective geometry $\mathfrak{G}$, and proved the following theorem.

Theorem 19. (Lyndon [10]) *For every projective geometry $\mathfrak{G}$, $\mathfrak{A}(\mathfrak{G}) \in \mathsf{RRA}$ iff $\mathfrak{G}$ can be embedded as a hyperplane in a projective geometry $\mathfrak{H}$, whose dimension is one more than the dimension of $\mathfrak{G}$.*

Using his construction, Lyndon obtained an integral nonrepresentable relation algebra with 8 atoms from a projective line with 6 points. The next reduction in size for the smallest known integral nonrepresentable relation algebra was accomplished by McKenzie.

Theorem 20. (McKenzie [17, 18]) *There are at least 60 integral relation algebras with 4 atoms, and one of them is not representable.*

Equation (M) is valid in RRA but fails in McKenzie's example. (This was not McKenzie's proof.)

$$\text{(M)}\qquad \begin{aligned}&x_{01} \cdot (x_{02} \cdot x_{03};x_{32});(x_{21} \cdot x_{24};x_{41})\\ &\le x_{03};\left((x_{30};x_{01} \cdot x_{32};x_{21});x_{14} \cdot x_{32};x_{24} \cdot x_{30};(x_{01};x_{14} \cdot x_{02};x_{24})\right);x_{41}.\end{aligned}$$

There can be no smaller example than Mckenzie's. Indeed, Lyndon [8] noted that there are exactly 13 integral relation algebras with 3 or fewer atoms and all of them are representable. It follows that an integral nonrepresentable relation algebra must have at least 4 atoms. The enumerations begun by Lyndon and McKenzie have been continued by others. Some results are given in the next theorem. To state it, we first need some definitions that are explained after the theorem. An element x of a relation algebra is **symmetric** if $\breve{x} = x$. Whenever $1 \le s \le a < \omega$ and $a - s$ is even, let $R(a, s)$ be the number of isomorphism types of relation algebras that have exactly a atoms and exactly s symmetric atoms, let

$$Q(a, s) = \frac{1}{6}(a-1)\left((a-1)^2 + 3s - 1\right)$$

and let

$$P(a, s) = 2^{\frac{1}{2}(a-s)}(s-1)!\left(\frac{1}{2}(a-s)\right)!.$$

The following theorem gives the values of $R(a, s)$ for various choices of a and s. They have been computed by hand and by computer. Lyndon computed the first four values (1, 2, 3, and 7), and S. Comer computed 37 and 65. Some others who have confirmed or added to these results are F. Backer, S. Givant, P. Jipsen, E. Lukacs, R. L. Kramer, R. McKenzie, U. Wostner, and myself.

Theorem 21.

a	s	$R(a,s)$	*atoms*	$Q(a,s)$	$P(a,s)$
1	1	1	1'	0	1
2	2	2	1'a	1	1
3	1	3	1'aǎ	2	2
3	3	7	1'ab	4	2
4	2	37	1'abb̆	7	2
4	4	65	1'abc	10	6
5	1	83	1'aǎbb̆	12	8
5	3	$1,316$	1'abcč	16	4
5	5	$3,013$	1'abcd	20	24
6	2	$47,865$	1'abb̆cč	25	8
6	4	$988,464$	1'abcdď	30	12
6	6	$3,849,920$	1'abcde	35	120

(1) *Exactly* 11 *of the* $37 = R(4,2)$ *are not representable.*
(2) *Exactly* 20 *of the* $65 = R(4,4)$ *are not representable.*
(3) *At least* 28 *of the* $83 = R(5,2)$ *are not representable.*

How many integral relation algebras are there whose atoms are 1', a, ǎ, b, b̆? In this case, $a = 5$, and only 1' is symmetric, so $s = 1$. There are $Q(5,1) = 12$ "cycles" from which to choose, and every set of cycles determines an algebra in NA that may, or may not, be a relation algebra. With probability approaching 1 as a increases, a randomly chosen set of cycles will produce a nonassociative relation algebra that is rigid, that is, one with no nontrivial automorphisms [13]. The number of potential automorphisms of an algebra having atoms 1', a, ǎ, b, b̆ is $P(5,1) = 8$. A randomly chosen set of cycles will therefore probably produce an algebra isomorphic to 7 other copies of itself that arise from 7 other sets of cycles. The expected number of isomorphism types of *potential* relation algebras having atoms 1', a, ǎ, b, b̆ is therefore

$$\frac{2^{Q(5,1)}}{P(5,1)} = \frac{2^{12}}{7} = 585.14\cdots .$$

How many of these are actually relation algebras? The next theorem shows that a randomly chosen set of cycles is a relation algebra with probability approaching 1 as a increases, so the expected number of isomorphism types of relation algebras having atoms 1', a, ǎ, b, b̆ is about 585, but the actual number is $R(5,1) = 83$.

If $1 \le s \le a < \omega$, $a - s$ is even, and $3 \le n$, let $B(n,a,s)$ be the number of isomorphism types of integral semi-associative relation algebras $\mathfrak{A}$ such that $\mathfrak{A}$ has a atoms, exactly s atoms are symmetric, and $B_n\mathfrak{A}$ is an n-dimensional relational basis for $\mathfrak{A}$ (so $\mathfrak{A} \in \mathsf{RA}_n$). Note that $B(4,a,s) = R(a,s)$ by theorem 10.

Theorem 22. [13], [16]

$$B(n,a,s) \approx \frac{2^{Q(a,s)}}{P(a,s)} = \frac{2^{\frac{1}{6}(a-1)((a-1)^2+3s-1)}}{(s-1)!\left(\frac{1}{2}(a-s)\right)!2^{\frac{1}{2}(a-s)}},$$

that is, for every real number $\varepsilon > 0$ *there is some* $N < \omega$ *such that if* $N < a \ge s \ge 1$ *and* $a - s$ *is even, then*

$$|1 - \frac{B(n,a,s)P(a,s)}{2^{Q(a,s)}}| < \varepsilon.$$

Let $3 \leq n < \omega$. It follows from the previous theorem that a randomly chosen integral nonassociative relation algebra $\mathfrak{A}$ with a atoms and s symmetric atoms will almost certainly (that is, with probability approaching 1 as a increases) be in RA_n. In fact, $B_n\mathfrak{A}$ will almost certainly be an n-dimensional relational basis for $\mathfrak{A}$. Let Σ be a finite set of equations that are valid in RRA. By theorem 14, there is some finite $n \geq 3$ such that all equations in Σ are valid in RA_n. Consequently, a randomly chosen finite integral relation algebra will almost certainly satisfy all the equations in Σ. It seems reasonable to conjecture that the same result would be obtained even if Σ contained *all* equations valid in RRA.

Open Problem. Prove that a randomly chosen finite integral relation algebra is almost certainly representable.

It is easy to show that (J), (L), and (M) are valid in RA_5, hence almost all finite integral relation algebras satisfy (J), (L), and (M). Nevertheless, none of these three can be deduced from the other two.

Theorem 23. [16] (J)*,* (L)*, and* (M) *are independent over* RA*.*

Each of (J), (L), and (M) fails in some relation algebra that satisfies the other two. Such algebras are scarcest for (L). In fact, the non-derivability of (L) from (M) and the axioms of RA is shown by only 2 out of the 115 integral RA's with 4 atoms, and the non-derivability of (L) from (M), (J), and the axioms of RA is shown by only 2 out of the 4527 integral RA's with no more than atoms. (J), (L), and (M) are all valid in RA_5, but because they are independent, no one of them is sufficient to axiomatize RA_5 relative to RA. This and other considerations led to the conjecture, proved by Hirsch and Hodkinson, that RA_5 is not finitely axiomatizable relative to RA, and, more generally, RA_{n+1} is not finitely axiomatizable relative to RA_n for $n \geq 4$.

Let $3 \leq n < \omega$, Every finite algebra $\mathfrak{A} \in \mathsf{NA}$ is isomorphic to its perfect extension: $\mathfrak{A} \cong \mathfrak{A}^+$. It follows from this, by theorem 13, that a finite algebra is in RA_n if and only if it has an n-dimensional relational basis. This gives an algorithm for determining whether or not a finite algebra $\mathfrak{A}$ is in RA_n: just check the subsets of $B_n\mathfrak{A}$ to see whether one of them is an n-dimensional relational basis. Therefore (*modulo* a proper formulation of the requisite definitions), the set of finite algebras in RA_n is recursive. If a finite algebra is not representable then, by theorem 14, it must fail to be in some RA_n. This gives an algorithm for enumerating the finite nonrepresentable algebras: generate the finite algebras in NA, check each one for membership in RA_3, RA_4, RA_5, *etc.*, and print the ones that fail to be in some RA_n. Instead of checking membership in RA_n, one can check whether each finite algebra satisfies the equations in some recursive (and necessarily infinite [19]) equational axiomatization of RRA, such as the one by Lyndon [9]. So the set of finite nonrepresentable algebras is recursively enumerable. Is it recursive? This natural question was open for more than 20 years before being answered in the negative by Hirsch and Hodkinson [5]. Long before these results were obtained, Tarski remarked to me that it would be a challenging task to develop a structure theory for finite relation algebras. He was right.

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