# 1. Peano Arithmetic (PA)

The language $L_\Omega$ of Peano arithmetic has a constant 0, a unary function symbol $S$, a binary function symbol $+$, binary function symbol $\cdot$, and the equality symbol $=$. There are two versions, both containing these axioms:

| | |
|---|---|
| Ax1 | $\neg(0 = Sx)$ |
| Ax2 | $Sx = Sy \rightarrow x = y$ |
| Ax3 | $x + 0 = x$ |
| Ax4 | $S(x + y) = x + Sy$ |
| Ax5 | $x \cdot 0 = 0$ |
| Ax6 | $x \cdot Sy = x \cdot y + x$ |

$\mathsf{PA}^2$ (**second-order Peano arithmetic**) has induction for all subsets $Q$. This can be stated as a sentence with a second-order quantifier "$\forall_Q$", where "$Q(x)$" means "$x$ is an element of $Q$".

$$\forall_Q(Q(0) \rightarrow (\forall_x(Q(x) \rightarrow Q(Sx)) \rightarrow \forall_x Q(x))). \tag{1}$$

A less formal statement of induction for a structure is that for all subsets $Q$ of the universe of the structure, if $0 \in Q$ and for all $x$, $x \in Q$ implies $Sx \in Q$, then $Q$ is the universe of the structure.

PA (**first-order Peano arithmetic**) replaces the single second-order statement of induction with axioms that assert infinitely many particular cases of induction, one for each *formula* $\psi(x)$ of $L_\Omega$:

$$\psi(0) \rightarrow (\forall_x(\psi(x) \rightarrow \psi(Sx)) \rightarrow \forall_x \psi(x)). \tag{2}$$

PA is a theory in $L_\Omega$, but $\mathsf{PA}^2$ is not a theory in $L_\Omega$.

In any structure $\mathfrak{A}$ for $L_\Omega$, the binary relation symbol $=$ is interpreted as the equality relation on the nonempty domain $A$, the binary function symbol $+$ is interpreted as the binary operation $+^{\mathfrak{A}}$ on $A$, the binary function symbol $\cdot$ is interpreted as the binary operation $\cdot^{\mathfrak{A}}$ on $A$, and the unary function symbol $S$ is interpreted as the function $S^{\mathfrak{A}}$ from $A$ to $A$.

The **standard model** is $\Omega$, which has domain $\omega = \{0, 1, 2, \cdots\}$ (the natural numbers), $S^\Omega$ is "+1" (the function that adds 1 to its input), $\cdot^\Omega$ is the binary operation $\cdot$ of multiplication of natural numbers, and $+^\Omega$ is the binary operation $+$ of addition of natural numbers.

A **nonstandard model** is any model of PA that is not isomorphic to the standard model.

Second-order $\mathsf{PA}^2$ is categorical: *any* two models are isomorphic. (Why?)

PA has only infinite models (Why? This follows from just Ax1 and Ax2).

PA has models of all infinite cardinalities (by the upward Löwenheim-Skolem-Tarski Theorem).

PA has $2^{\aleph_0}$ pairwise non-isomorphic models of cardinality $\aleph_0$.

In the proofs of theorems 1–24, we assume that $\mathfrak{A} = \langle A, +, \cdot, S, 0\rangle$ is an arbitrary model of axioms Ax1–Ax6 and (2). Theorems 1–24 show that various other formulas are satisfied in $\mathfrak{A}$ for all possible values of the variables. By the Completeness Theorem, it follows that those formulas are provable consequences of PA.

Prop1: PA $\vdash x = 0 + x$.

*Proof.* Let $\psi(x)$ be $x = 0 + x$.

$\psi(0)$ holds because $0 = 0 + 0$ by Ax3.

Show: $\psi(x) \to \psi(Sx)$.

Hyp: $x = 0 + x$. Show: $Sx = 0 + Sx$.

$$\begin{aligned} Sx &= S(0 + x) && \text{Hyp} \\ &= 0 + Sx && \text{Ax4} \end{aligned}$$

□

Prop2: PA $\vdash Sx + y = S(x + y)$.

*Proof.* Let $\psi(y)$ be $Sx + y = S(x + y)$.

$\psi(0)$ holds because

$$\begin{aligned} Sx + 0 &= Sx && \text{Ax3} \\ &= S(x + 0) && \text{Ax3} \end{aligned}$$

Show: $\psi(y) \to \psi(Sy)$.

Hyp: $Sx + y = S(x + y)$. Show: $Sx + Sy = S(x + Sy)$.

$$\begin{aligned} Sx + Sy &= S(Sx + y) && \text{Ax4} \\ &= SS(x + y) && \text{Hyp} \\ &= S(x + Sy) && \text{Ax4} \end{aligned}$$

□

Prop3: PA $\vdash x + y = y + x$.

*Proof.* Let $\psi(y)$ be $x + y = y + x$.

$\psi(0)$ holds because

$$\begin{aligned} x + 0 &= x && \text{Ax3} \\ &= 0 + x && \text{Prop1} \end{aligned}$$

Show: PA $\vdash \psi(y) \to \psi(Sy)$.

Hyp: $x + y = y + x$. Show: $x + Sy = Sy + x$.

$$\begin{aligned} x + Sy &= S(x + y) && \text{Ax4} \end{aligned}$$

$$
\begin{aligned}
&= S(y+x) && \text{Hyp}\\
&= Sy + x && \textsf{Prop2}
\end{aligned}
$$

□

Prop4: PA $\vdash x + (y + z) = (x + y) + z$.

*Proof.* Let $\psi(z)$ be $x + (y + z) = (x + y) + z$.

$\psi(0)$ holds because

$$
\begin{aligned}
x + (y + 0) &= x + y && \textsf{Ax3}\\
&= (x + y) + 0 && \textsf{Ax3}
\end{aligned}
$$

Show: $\psi(z) \to \psi(Sz)$.

Hyp: $x + (y + z) = (x + y) + z$. Show: $x + (y + Sz) = (x + y) + Sz$.

$$
\begin{aligned}
x + (y + Sz) &= x + S(y + z) && \textsf{Ax4}\\
&= S(x + (y + z)) && \textsf{Ax4}\\
&= S((x + y) + z) && \text{Hyp}\\
&= (x + y) + Sz && \textsf{Ax4}
\end{aligned}
$$

□

Prop5: PA $\vdash 0 \cdot x = 0$.

*Proof.* Let $\psi(x)$ be $0 \cdot x = 0$.

$\psi(0)$ holds because $0 \cdot 0 = 0$ by Ax5.

Show: $\psi(x) \to \psi(Sx)$.

Hyp: $0 \cdot x = 0$. Show: $0 \cdot Sx = 0$.

$$
\begin{aligned}
0 \cdot Sx &= 0 \cdot x + 0 && \textsf{Ax6}\\
&= 0 + 0 && \text{Hyp}\\
&= 0 && \textsf{Ax3}
\end{aligned}
$$

□

Prop6: PA $\vdash Sx \cdot y = x \cdot y + y$.

*Proof.* Let $\psi(y)$ be $Sx \cdot y = x \cdot y + y$.

$\psi(0)$ holds because

$$
\begin{aligned}
Sx \cdot 0 &= 0 && \textsf{Ax5}\\
&= 0 + 0 && \textsf{Ax3}\\
&= x \cdot 0 + 0 && \textsf{Ax5}
\end{aligned}
$$

Show: $\psi(y) \to \psi(Sy)$.

Hyp: $Sx \cdot y = x \cdot y + y$. Show: $Sx \cdot Sy = x \cdot Sy + Sy$.

$$\begin{aligned} Sx \cdot Sy &= Sx \cdot y + Sx && \text{Ax6} \\ &= (x \cdot y + y) + Sx && \text{Hyp} \\ &= x \cdot y + (y + Sx) && \text{Prop4} \\ &= x \cdot y + S(y + x) && \text{Ax4} \\ &= x \cdot y + S(x + y) && \text{Prop3} \\ &= x \cdot y + (x + Sy) && \text{Ax4} \\ &= (x \cdot y + x) + Sy && \text{Prop4} \\ &= x \cdot Sy + Sy && \text{Ax6} \end{aligned}$$

□

Prop7: PA $\vdash x \cdot y = y \cdot x$.

*Proof.* Let $\psi(y)$ be $x \cdot y = y \cdot x$.

$\psi(0)$ holds because

$$\begin{aligned} x \cdot 0 &= x && \text{Ax5} \\ &= 0 \cdot x && \text{Prop5} \end{aligned}$$

Show: PA $\vdash \psi(y) \to \psi(Sy)$.

Hyp: $x \cdot y = y \cdot x$. Show: $x \cdot Sy = Sy \cdot x$.

$$\begin{aligned} x \cdot Sy &= x \cdot y + x && \text{Ax6} \\ &= y \cdot x + x && \text{Hyp} \\ &= Sy \cdot x && \text{Prop6} \end{aligned}$$

□

Prop8: PA $\vdash x \cdot (y + z) = x \cdot y + x \cdot z$.

*Proof.* Let $\psi(z)$ be $x \cdot (y + z) = x \cdot y + x \cdot z$.

$\psi(0)$ holds because

$$\begin{aligned} x \cdot (y + 0) &= x \cdot y && \text{Ax3} \\ &= x \cdot y + 0 && \text{Ax3} \\ &= x \cdot y + x \cdot 0 && \text{Ax5} \end{aligned}$$

Show: $\psi(z) \to \psi(Sz)$.

Hyp: $x \cdot (y + z) = x \cdot y + x \cdot z$. Show: $x \cdot (y + Sz) = x \cdot y + x \cdot Sz$.

$$\begin{aligned} x \cdot (y + Sz) &= x \cdot (S(y + z)) && \text{Ax4} \\ &= x \cdot (y + z) + x && \text{Ax6} \end{aligned}$$

$$
\begin{aligned}
&= (x \cdot y + x \cdot z) + x && \text{Hyp.} \\
&= x \cdot y + (x \cdot z + x) && \mathsf{Prop4} \\
&= x \cdot y + x \cdot Sz && \mathsf{Ax6}
\end{aligned}
$$

□

Prop9: PA $\vdash (x + y) \cdot z = x \cdot z + y \cdot z$.

*Proof.* By Prop7 and Prop8. □

Prop10: PA $\vdash (x \cdot y) \cdot z = x \cdot (y \cdot z)$.

*Proof.* Let $\psi(z)$ be $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

$\psi(0)$ holds because

$$
\begin{aligned}
(x \cdot y) \cdot 0 &= 0 && \mathsf{Ax5} \\
&= x \cdot 0 && \mathsf{Ax5} \\
&= x \cdot (y \cdot 0) && \mathsf{Ax5}
\end{aligned}
$$

Show: $\psi(z) \to \psi(Sz)$.

Hyp: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Show: $(x \cdot y) \cdot Sz = x \cdot (y \cdot Sz)$.

$$
\begin{aligned}
(x \cdot y) \cdot Sz &= (x \cdot y) \cdot z + x \cdot y && \mathsf{Ax6} \\
&= x \cdot (y \cdot z) + x \cdot y && \text{Hyp.} \\
&= x \cdot (y \cdot z + y) && \mathsf{Prop8} \\
&= x \cdot (y \cdot Sz) && \mathsf{Ax5}
\end{aligned}
$$

□

Prop11: PA $\vdash x + y = x + z \to y = z$.

*Proof.* Let $\psi(x)$ be $x + y = x + z \to y = z$.

$\psi(0)$ holds because

| | | |
|---|---|---|
| 1. | $0 + y = 0 + z$ | hyp. |
| 1. | $y = z$ | 1., Prop1 |

Show: PA $\vdash \psi(x) \to \psi(Sx)$.

Hyp: $x + y = x + z \to y = z$. Show: $Sx + y = Sx + z \to y = z$.

| | | |
|---|---|---|
| 1. | $Sx + y = Sx + z$ | hyp. |
| 2. | $S(x + y) = S(x + z)$ | Prop2 |
| 3. | $x + y = x + z$ | Ax2 |
| 4. | $y = z$ | Hyp. |

□

Prop12: PA $\vdash x \cdot S0 = x$ and $x + S0 = Sx$

*Proof.*

$$\begin{aligned} x \cdot S0 &= x \cdot 0 + x && \text{Ax6} \\ &= 0 + x && \text{Ax5} \\ &= x && \text{Prop1} \\ x + S0 &= S(x+0) && \text{Ax4} \\ &= Sx && \text{Ax3} \end{aligned}$$

□

Prop13: PA $\vdash x + y = 0 \to x = 0 \wedge y = 0$.

*Proof.* Let $\psi(y)$ be $x + y = 0 \to x = 0 \wedge y = 0$.

$\psi(0)$ holds because

| | | |
|---|---|---|
| 1. | $x + 0 = 0$ | hyp. |
| 2. | $x = 0$ | 1., Ax3 |
| 3. | $0 = 0$ | FOLE |
| 4. | $x = 0 \wedge 0 = 0$ | 2., 3. |

Show: $\psi(y) \to \psi(Sy)$.

Hyp: $x + y = 0 \to x = 0 \wedge y = 0$. Show: $x + Sy = 0 \to x = 0 \wedge Sy = 0$.

| | | |
|---|---|---|
| 1. | $x + Sy = 0$ | hyp. |
| 2. | $S(x + y) = 0$ | 1., Ax4 |
| 3. | $\neg S(x + y) = 0$ | Ax1 |
| 4. | $\neg(x + Sy = 0)$ | 1.–3. |
| 5. | $x + Sy = 0 \to x = 0 \wedge Sy = 0$ | 4., PC |

□

Prop14: PA $\vdash x \cdot y = 0 \to x = 0 \vee y = 0$.

*Proof.* Let $\psi(y)$ be $x \cdot y = 0 \to x = 0 \vee y = 0$.

$\psi(0)$ is $x \cdot 0 = 0 \to x = 0 \vee 0 = 0$. Hypothesis is true by Ax5, conclusion is true since $0 = 0$, so $\psi(0)$ holds.

Show: $\psi(y) \to \psi(Sy)$.

Hyp: $x \cdot y = 0 \to x = 0 \vee y = 0$. Show: $x \cdot Sy = 0 \to x = 0 \vee Sy = 0$.

| | | |
|---|---|---|
| 1. | $x \cdot Sy = 0$ | hyp |
| 2. | $x \cdot y + x = 0$ | 1., Ax6 |
| 3. | $x \cdot y = 0 \wedge x = 0$ | 2., Prop13 |

| | | |
|---|---|---|
| 4. | $x = 0 \wedge Sy = 0$ | 3., PC |

□

Prop15: $\mathsf{PA} \vdash z = 0 \vee \exists_w (z = Sw)$.

*Proof.* Let $\psi(z)$ be $z = 0 \vee \exists_w (z = Sw)$.

$\psi(0)$ is $0 = 0 \vee \exists_w (0 = Sw)$, which holds by PC because $0 = 0$.

Show: $\psi(z) \to \psi(Sz)$.

Hyp: $z = 0 \vee \exists_w (z = Sw)$. Show: $Sz = 0 \vee \exists_w (Sz = Sw)$.

By Ax1, $\neg(Sz = 0)$, so $\psi(Sz)$ is equivalent to $\exists_w (Sz = Sw)$, which is true. □

Prop16: $\mathsf{PA} \vdash Sx \cdot y = Sx \cdot z \to y = z$.

*Proof.* Let $\psi(y)$ be $\forall_z (Sx \cdot y = Sx \cdot z \to y = z)$.

$\Psi(0)$ holds because for arbitrary $z$ we have

| | | |
|---|---|---|
| 1. | $Sx \cdot 0 = Sx \cdot z$ | hyp |
| 2. | $0 = Sx \cdot z$ | Ax5 |
| 3. | $0 = Sx \vee 0 = z$ | 2., Prop14 |
| 4. | $\neg(0 = Sx)$ | Ax1 |
| 5. | $0 = z$ | 3., 4., PC |

Show: $\mathsf{PA} \vdash \psi(y) \to \psi(Sy)$.

Hyp: $\forall_z (Sx \cdot y = Sx \cdot z \to y = z)$. Show: $\forall_z (Sx \cdot Sy = Sx \cdot z \to Sy = z)$.

| | | |
|---|---|---|
| 1. | $Sx \cdot Sy = Sx \cdot z$ | hyp. |
| 2. | $\neg(Sx = 0) \wedge \neg(Sy = 0)$ | Ax1 |
| 3. | $\neg(Sx = 0 \vee Sy = 0)$ | 2., DeM |
| 4. | $Sx \cdot \neg(Sy = 0)$ | 3., Prop14 (contrapositive) |
| 5. | $Sx \cdot \neg(z = 0)$ | 4., 1. |
| 6. | $\neg(z = 0)$ | Ax5 (contrapositive) |
| 7. | $z = Sw$, for some $w$ | 6., Prop15 |
| 8. | $Sx \cdot Sy = Sx \cdot Sw$ | 7., 1. |
| 9. | $Sx \cdot y + Sx = Sx \cdot w + Sx$ | 8., Ax6 |
| 10. | $Sx \cdot y = Sx \cdot w$ | 9., Prop11 |
| 11. | $Sx \cdot y = Sx \cdot w \to y = w$ | Hyp. with $w$ for $z$ |
| 12. | $y = w$ | 10., 11., PC |
| 13. | $Sy = Sw$ | 12. |
| 14. | $Sy = z$ | 7., 13. |

□

Next we define certain terms called **numerals**. There is a numeral for each natural number $n \in \omega$. The numeral $\overline{0}$ is just the constant 0. For every positive $n \in \omega$, the numeral $\overline{n}$ is the term $\overbrace{SS\cdots S}^{n}0$. For example, $\overline{1} = S0$, $\overline{2} = SS0$, $\overline{3} = SSS0$, $\overline{4} = SSSS0$.

Prop17: $\mathsf{PA} \vdash x + \overline{1} = Sx$

*Proof.*

| | |
|---|---|
| $x + \overline{1} = x + S0$ | |
| $= Sx$ | Prop12 |

□

Prop18: $\mathsf{PA} \vdash x \cdot \overline{1} = x$

*Proof.*

| | |
|---|---|
| $x \cdot \overline{1} = x \cdot S0$ | def $\overline{1}$ |
| $= x$ | Prop12 |

□

Prop19: $\mathsf{PA} \vdash x \cdot \overline{2} = x + x$

*Proof.*

| | |
|---|---|
| $x \cdot \overline{2} = x \cdot SS0$ | def $\overline{2}$ |
| $= x \cdot S0 + x$ | Ax6 |
| $= x + x$ | Prop12 |

□

Prop20: $\mathsf{PA} \vdash \neg(x = 0) \rightarrow (y \cdot x = 0 \rightarrow y = 0)$

*Proof.*

| | | |
|---|---|---|
| 1. | $\neg(x = 0)$ | hyp. |
| 2. | $y \cdot x = 0$ | hyp. |
| 3. | $\exists_w(x = Sw)$ | 1., Prop15 |
| 4. | $x = Sw$ | $\exists$-rule |
| 5. | $y \cdot Sw = 0$ | 2., 4. |
| 6. | $0 = 0 \cdot Sw$ | Prop5 |
| 7. | $y \cdot Sw = 0 \cdot Sw$ | 5., 6. |
| 8. | $Sw \cdot y = Sw \cdot 0$ | Prop7 |
| 9. | $y = 0$ | 8., Prop16 |
| 10. | $y \cdot x = 0 \rightarrow y = 0$ | 2.—9. |
| 11. | $\neg(x = 0) \rightarrow (y \cdot x = 0 \rightarrow y = 0)$ | 1.—10. |

□

Prop21: $\mathsf{PA} \vdash x + y = \overline{1} \rightarrow (x = 0 \vee y = 0)$

*Proof.*

| | | |
|---|---|---|
| 1. | $x + y = \overline{1}$ | hyp. |
| 2. | $x + y = S0$ | def $\overline{1}$ |
| 3. | $\neg(x = 0)$ | hyp. |
| 4. | $\exists_w (x = Sw)$ | Prop15 |
| 5. | $x = Sw$ | $\exists$-rule |
| 6. | $Sw + y = S0$ | 2., 5. |
| 7. | $S(w + y) = S0$ | Prop2 |
| 8. | $w + y = 0$ | Ax2, MP |
| 9. | $w = 0 \wedge y = 0$ | Prop13 |
| 10. | $y = 0$ | PC |
| 11. | $\neg(x = 0) \rightarrow y = 0$ | 3.—10. |
| 12. | $x = 0 \vee y = 0$ | 11., PC |

□

Prop22: $\mathsf{PA} \vdash x + y = \overline{1} \rightarrow ((x = 0 \wedge y = \overline{1}) \vee (x = \overline{1} \wedge y = 0))$

*Proof.* Note that $x + y = \overline{1} \rightarrow (x = 0 \rightarrow y = \overline{1})$ by Prop1, and $x + y = \overline{1} \rightarrow (y = 0 \rightarrow x = \overline{1})$ by Ax1, so the result follows by Prop21 and PC. □

Prop23: $\mathsf{PA} \vdash x \cdot y = \overline{1} \rightarrow (x = \overline{1} \wedge y = \overline{1})$

*Proof.*

| | | |
|---|---|---|
| 1. | $x \cdot y = \overline{1}$ | hyp. |
| 2. | $y = 0$ | hyp. |
| 3. | $x \cdot 0 = S0$ | 1., 2., def $\overline{1}$ |
| 4. | $0 = S0$ | Ax5 |
| 5. | contradiction | 4., Ax1 |
| 6. | $\neg(y = 0)$ | 2.—5. |
| 7. | $\exists_w (y = Sw)$ | Prop15 |
| 8. | $y = Sw$ | for some $w$, $\exists$-rule |
| 9. | $x \cdot Sw = S0$ | 1., 8. |
| 10. | $x \cdot w + x = S0$ | Ax6, def $\overline{1}$ |
| 11. | $x \cdot w = 0 \vee x = 0$ | Prop21 |
| 12. | $x = 0$ | hyp. |

| | | |
|---|---|---|
| 13. | $0 \cdot y = S0$ | 1., 12., def $\overline{1}$ |
| 14. | $0 = S0$ | Prop5 |
| 15. | contradiction | 14., Ax1 |
| 16. | $\neg(x = 0)$ | 12.—15. |
| 17. | $x \cdot w = 0$ | 16., 11. |
| 18. | $0 + x = S0$ | 10., 17. |
| 19. | $x = S0$ | Prop1 |
| 20. | $x = \overline{1}$ | def $\overline{1}$ |
| 21. | $S0 \cdot y = \overline{1}$ | 1., 19. |
| 22. | $y \cdot S0 = \overline{1}$ | Prop7 |
| 23. | $y = \overline{1}$ | Prop12 |
| 24. | $x = \overline{1} \wedge y = \overline{1}$ | 20., 23., Adj. |

□

Prop24: PA $\vdash \neg(x = 0) \rightarrow (\neg(x = \overline{1}) \rightarrow \exists_z(x = SSz))$.

*Proof.*

| | | |
|---|---|---|
| 1. | $\neg(x = 0)$ | hyp. |
| 2. | $\neg(x = \overline{1})$ | hyp. |
| 3. | $\exists_w(x = Sw)$ | 1., Prop15 |
| 4. | $x = Sw$ | for some $w$, $\exists$-rule |
| 5. | $\neg(x = S0)$ | 2., def $\overline{1}$ |
| 6. | $w = 0$ | hyp. |
| 7. | $x = S0$ | 4., 6. |
| 8. | contradiction | 5., 7. |
| 9. | $\neg(w = 0)$ | 6.—8. |
| 10. | $\exists_z(w = Sz)$ | 9., Th15 |
| 11. | $w = Sz$ | for some $z$, $\exists$-rule |
| 12. | $x = SSz$ | 4., 11. |
| 13. | $\exists_z(x = SSz)$ | 12., FOL |
| 14. | $\neg(x = \overline{1}) \rightarrow \exists_z(x = SSz)$ | 2.—13. |
| 15. | $\neg(x = 0) \rightarrow (\neg(x = \overline{1}) \rightarrow \exists_z(x = SSz))$ | 1.—14. |

□

| | |
|---|---|
| Prop1 | PA $\vdash x = 0 + x$ |
| Prop2 | PA $\vdash Sx + y = S(x + y)$ |
| Prop3 | PA $\vdash x + y = y + x$ |

| | |
|---|---|
| Prop4 | $\mathsf{PA} \vdash x + (y + z) = (x + y) + z$ |
| Prop5 | $\mathsf{PA} \vdash 0 \cdot x = 0$ |
| Prop6 | $\mathsf{PA} \vdash Sx \cdot y = x \cdot y + y$ |
| Prop7 | $\mathsf{PA} \vdash x \cdot y = y \cdot x$ |
| Prop8 | $\mathsf{PA} \vdash x \cdot (y + z) = x \cdot y + x \cdot z$ |
| Prop9 | $\mathsf{PA} \vdash (x + y) \cdot z = x \cdot z + y \cdot z$ |
| Prop10 | $\mathsf{PA} \vdash (x \cdot y) \cdot z = x \cdot (y \cdot z)$ |
| Prop11 | $\mathsf{PA} \vdash x + y = x + z \rightarrow y = z$ |
| Prop12 | $\mathsf{PA} \vdash x \cdot S0 = x$ and $x + S0 = Sx$ |
| Prop13 | $\mathsf{PA} \vdash x + y = 0 \rightarrow x = 0 \wedge y = 0$ |
| Prop14 | $\mathsf{PA} \vdash x \cdot y = 0 \rightarrow x = 0 \vee y = 0$ |
| Prop15 | $\mathsf{PA} \vdash z = 0 \vee \exists_w (z = Sw)$ |
| Prop16 | $\mathsf{PA} \vdash Sx \cdot y = Sx \cdot z \rightarrow y = z$ |
| Prop17 | $\mathsf{PA} \vdash x + \overline{1} = Sx$ |
| Prop18 | $\mathsf{PA} \vdash x \cdot \overline{1} = x$ |
| Prop19 | $\mathsf{PA} \vdash x \cdot \overline{2} = x + x$ |
| Prop20 | $\mathsf{PA} \vdash \neg(x = 0) \rightarrow (y \cdot x = 0 \rightarrow y = 0)$ |
| Prop21 | $\mathsf{PA} \vdash x + y = \overline{1} \rightarrow (x = 0 \vee y = 0)$ |
| Prop22 | $\mathsf{PA} \vdash x + y = \overline{1} \rightarrow ((x = 0 \wedge y = \overline{1}) \vee (x = \overline{1} \wedge y = 0))$ |
| Prop23 | $\mathsf{PA} \vdash x \cdot y = \overline{1} \rightarrow (x = \overline{1} \wedge y = \overline{1})$ |
| Prop24 | $\mathsf{PA} \vdash \neg(x = 0) \rightarrow (\neg(x = \overline{1}) \rightarrow \exists_z (x = SSz))$ |

**Theorem 1.** If $m, n \in \omega$ and $m < n$ then $\mathsf{PA} \vdash \neg(\overline{m} = \overline{n})$.

*Proof.* Let $\Delta = \{m : m \in \omega,\ \text{if } m < n \in \omega \text{ then } \mathsf{PA} \vdash \neg(\overline{m} = \overline{n})\}$.

First we show that $0 \in \Delta$. To prove this we must show that if $0 < n \in \omega$ then $\mathsf{PA} \vdash \neg(\overline{0} = \overline{n})\}$. Suppose $0 < n \in \omega$. Then $\overline{n} = S\overline{n-1}$ since $n \geq 1$, and $\neg(0 = S\overline{n-1})$ is an instance of Ax1, so we have $\mathsf{PA} \vdash \neg(0 = S\overline{n-1})$. But 0 is the numeral $\overline{0}$, so $\mathsf{PA} \vdash \neg(\overline{0} = \overline{n})$.

Next, suppose $m \in \Delta$. We will prove from this assumption that $m + 1 \in \Delta$. Suppose $m + 1 < n$. Then $m < n - 1$, but $m \in \Delta$, so $\mathsf{PA} \vdash \neg(\overline{m} = \overline{n-1})$, hence $\mathsf{PA} \vdash \neg(S\overline{m} = S\overline{n-1})$ by the contrapositive of Ax2, hence $\mathsf{PA} \vdash \neg(\overline{m+1} = \overline{n})$, as desired.

Since $0 \in \Delta$ and $m + 1 \in \Delta$ whenever $m \in \Delta$, it follows by induction that $\Delta = \omega$. □

**Corollary 1.** All models of PA are infinite.

*Proof.* In any model of PA, the infinitely many numerals $\overline{n}$ for $n \in \omega$ are interpreted as distinct elements by Theorem 1, hence the model is infinite. □

**Corollary 2.** $\mathsf{PA}$ has models of all infinite cardinalities.

*Proof.* By the Upward Löwenheim-Skolem theorem, any first-order theory with infinite models has models of all infinite cardinalities. $\mathsf{PA}$ is such a theory by the Corollary 1. □

**Theorem 2.** $\mathsf{PA} \vdash \overline{m+n} = \overline{m} + \overline{n}$

*Proof.* Let

$$\Delta := \{n : n \in \omega, (\forall m \in \omega)(\mathsf{PA} \vdash \overline{m} + \overline{n} = \overline{m+n})\}$$

Claim: $0 \in \Delta$. To prove this we must show that for all $m \in \omega$, $\mathsf{PA} \vdash \overline{m} + \overline{0} = \overline{m+0}$. Since $\overline{0} = 0$ and $\overline{m+0}$, we need only show $\mathsf{PA} \vdash \overline{m} + 0 = \overline{m}$, but this is true by $\mathsf{Ax6}$.

Claim: If $n \in \Delta$ then $n+1 \in \Delta$. Suppose $n \in \Delta$. To prove $n+1 \in \Delta$, suppose $m \in \omega$. We must show $\mathsf{PA} \vdash \overline{m} + \overline{n+1} = \overline{m+n+1}$.

The assumption $n \in \Delta$, applied to $m+1$, implies that $\mathsf{PA} \vdash \overline{m+1} + \overline{n} = \overline{m+1+n}$. Hence $\mathsf{PA} \vdash S\overline{m} + \overline{n} = \overline{m+n+1}$ by the definition of numeral, so $\mathsf{PA} \vdash S(\overline{m} + \overline{n}) = \overline{m+n+1}$ by $\mathsf{Prop3}$, so $\mathsf{PA} \vdash \overline{m} + S(\overline{n}) = \overline{m+n+1}$ by $\mathsf{Ax3}$, so $\mathsf{PA} \vdash \overline{m} + \overline{n+1} = \overline{m+n+1}$ by $\mathsf{Ax3}$, by definition of numeral. □

**Theorem 3.** $\mathsf{PA} \vdash \overline{mn} \neq \overline{m} \cdot \overline{n}$.

*Proof.* Let

$$\Delta := \{n : n \in \omega, (\forall m \in \omega)(\mathsf{PA} \vdash \overline{m} \cdot \overline{n} = \overline{mn})\}$$

Claim: $0 \in \Delta$. To prove this we must show that $\mathsf{PA} \vdash \overline{m} \cdot \overline{0} = \overline{m \cdot 0}$ for all $m \in \omega$. But $\overline{0}$ and $\overline{m \cdot 0}$ are both the constant 0, so we need to show $\mathsf{PA} \vdash \overline{m} \cdot 0 = 0$, which is true by $\mathsf{Ax5}$.

Claim: If $n \in \Delta$ then $n+1 \in \Delta$. Assume $n \in \Delta$. To establish $n+1 \in \Delta$, we must show $\mathsf{PA} \vdash \overline{m} \cdot \overline{n} = \overline{mn}$ for all $m \in \omega$. Let $m \in \omega$. From $n \in \Delta$ we get $\mathsf{PA} \vdash \overline{m} \cdot \overline{n} = \overline{mn}$. Hence $\mathsf{PA} \vdash \overline{m} \cdot \overline{n} + \overline{m} = \overline{mn} + \overline{m}$ by first-order logic. Then, $\mathsf{PA} \vdash \overline{m} \cdot \overline{n} + \overline{m} = \overline{mn+m}$ by the previous theorem, hence $\mathsf{PA} \vdash \overline{m} \cdot S\overline{n} = \overline{mn+m}$ by $\mathsf{Ax6}$. But $S\overline{n}$ is $\overline{n+1}$ and $\overline{mn+m}$ is $\overline{m(n+1)}$, so we have $\mathsf{PA} \vdash \overline{m} \cdot \overline{n+1} = \overline{m(n+1)}$, as desired. □

**Corollary 3.** Every model of $\mathsf{PA}$ contains a substructure isomorphic to the standard model.

*Proof.* In any model $\mathfrak{A}$ of $\mathsf{PA}$, the subset $\{\overline{n}^{\mathfrak{A}} : n \in \omega\}$ is closed under the operations of $\mathfrak{A}$ and is isomorphic to the standard model by Theorems 2 and 3. (Details omitted.) □

**Corollary 4.** Nonstandard countable models of $\mathsf{PA}$ exist.

*Proof.* Expand the language of $\mathsf{PA}$ by adding a new constant $c$. Let $\Sigma$ be the union of $\mathsf{PA}$ with $\{\neg(c = \overline{n}) : n \in \omega\}$. Then every finite subset $\Sigma_0$ of $\Sigma$ has a model, because the standard model has infinitely many elements while only finitely many inequalities appear in $\Sigma_0$, so it is possible to assign $c$ to a element of the standard model that is distinct from the interpretations of the finitely many numerals that occur in $\Sigma_0$. By the Compactness Theorem, $\Sigma$ has a model $\mathfrak{A}$, whose restriction to the language of $\mathsf{PA}$ is a model of $\mathsf{PA}$ with some element (the interpretation of $c$) that does not occur in the substructure of $\mathfrak{A}$ isomorphic to the standard model, hence $\mathfrak{A}$ is nonstandard. (Again, details are omitted.) $\square$

**Definitions.** $x < y$ is the formula $\exists_z(\neg(z = 0) \wedge x + z = y)$, where $z$ is the first variable distinct from $x$ and $y$. $x \leq y$ is the formula $\exists_z(x + z = y)$, where $z$ is the first variable distinct from $x$ and $y$. For terms $t, s$, $t < s$ is the formula $\exists_z(\neg(z = 0)\wedge t+z = s)$, and $t \leq s$ is the formula $\exists_z(t + z = s)$, where $z$ is the first variable that does not occur in $t$ or $s$.

**Theorem 4.**

$$\mathsf{PA} \vdash \neg(t < t) \tag{3}$$
$$\mathsf{PA} \vdash r < s \to (s < t \to r < t) \tag{4}$$
$$\mathsf{PA} \vdash r < s \to \neg(r < s) \tag{5}$$
$$\mathsf{PA} \vdash r < s \to r + t < s + t \tag{6}$$
$$\mathsf{PA} \vdash t \leq t \tag{7}$$
$$\mathsf{PA} \vdash r \leq s \to (s \leq t \to r \leq t) \tag{8}$$
$$\mathsf{PA} \vdash r \leq s \to r + t \leq s + t \tag{9}$$
$$\mathsf{PA} \vdash r \leq s \to (s < t \to r < t) \tag{10}$$
$$\mathsf{PA} \vdash 0 \leq t \tag{11}$$
$$\mathsf{PA} \vdash 0 < St \tag{12}$$
$$\mathsf{PA} \vdash r < t \to Sr \leq t \tag{13}$$
$$\mathsf{PA} \vdash r \leq t \to r < St \tag{14}$$
$$\mathsf{PA} \vdash t < St \tag{15}$$
$$\mathsf{PA} \vdash \overline{n} < \overline{n + 1} \tag{16}$$
$$\mathsf{PA} \vdash r \neq t \to r < t \vee t < r \tag{17}$$
$$\mathsf{PA} \vdash r \leq t \vee t \leq r \tag{18}$$
$$\mathsf{PA} \vdash t \leq t + r \tag{19}$$
$$\mathsf{PA} \vdash \neg(r = 0) \to t < t + r \tag{20}$$
$$\mathsf{PA} \vdash \neg(r = 0) \to t \leq t \cdot r \tag{21}$$
$$\mathsf{PA} \vdash \neg(t = 0) \to 0 < t \tag{22}$$
$$\mathsf{PA} \vdash 0 < r \to (0 < t \to 0 < r \cdot t) \tag{23}$$
$$\mathsf{PA} \vdash \neg(r = 0) \to (\overline{1} < t \to r < t \cdot r) \tag{24}$$
$$\mathsf{PA} \vdash \neg(r = 0) \to (s < t \to s \cdot r < t \cdot r) \tag{25}$$
$$\mathsf{PA} \vdash \neg(r = 0) \to (s \leq t \to s \cdot r \leq t \cdot r) \tag{26}$$
$$\mathsf{PA} \vdash \neg(t < 0) \tag{27}$$
$$\mathsf{PA} \vdash r \leq t \wedge t \leq r \to t = r \tag{28}$$

**Theorem 5.** For every $n \in \omega$,

$$\mathsf{PA} \vdash x = 0 \vee \cdots \vee x = \overline{n} \leftrightarrow x \leq \overline{n}$$

**Theorem 6.** For every $n \in \omega$ and every formula $\varphi$,

$$\mathsf{PA} \vdash \varphi(0) \vee \cdots \vee \varphi(\overline{n}) \leftrightarrow \exists_x (x \leq \overline{n} \wedge \varphi(x))$$

**Theorem 7.** Complete induction (for formulas $\psi$) is provable in $\mathsf{PA}$:

$$\mathsf{PA} \vdash (\forall_y (y < x \rightarrow \psi(y)) \rightarrow \psi(x)) \rightarrow \forall_x \psi(x).$$

**Theorem 32.** The least-number principle (for formulas $\psi$) is provable in $\mathsf{PA}$:

$$\mathsf{PA} \vdash \exists_x \psi(x) \rightarrow \exists_m (\psi(m) \wedge \forall_x (x < m \rightarrow \neg \psi(x))).$$

**Definition.** For terms $t$ and $s$, let $t|s$ be the formula $\exists_z (t \cdot z = s)$ where $z$ is the first variable not occurring in $t$ or $s$.

Theorems of $\mathsf{PA}$ about divisibility.

**Theorem 33.** $\mathsf{PA}$ proves a sentence that asserts the existence and uniqueness of quotient and remainder. Existence:

$$\forall_q \forall_n \exists_d \exists_r (n = q \cdot d + r \wedge r < d)$$

Existence and uniqueness:

$$\forall_q \forall_n \exists_d \exists_r (n = q \cdot d + r \wedge r < d \wedge \forall_e \forall_s (n = q \cdot e + s \wedge s < e \rightarrow d = e \wedge r = s))$$