Homework #9, due 11/4/09 = **4.4.1**, **4.4.6**, **4.4.8(a)(b)**

4.4.1 If $\sigma \in \mathrm{Aut}(G)$ and φ_g is conjugation by g prove $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)}$. Deduce that $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$.

To show $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)}$ it is enough to note that, for every $x \in G$,

$$\begin{aligned}
(\sigma\varphi_g\sigma^{-1})(x) &= \sigma(\varphi_g(\sigma^{-1}(x))) \\
&= \sigma(g\sigma^{-1}(x)g^{-1}) \\
&= \sigma(g)\sigma(\sigma^{-1}(x))\sigma(g)^{-1} && \sigma \text{ is an automorphism} \\
&= \sigma(g)x\sigma(g)^{-1} \\
&= \varphi_{\sigma(g)}(x)
\end{aligned}$$

To prove $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$ it is enough to show that if an inner automorphism $\varphi_g \in \mathrm{Inn}(G)$ is conjugated by an automorphism $\sigma \in \mathrm{Aut}(G)$, the result is again an inner automorphism of G. The calculation above shows that the conjugate of the inner automorphism φ_g by an automorphism σ is $\varphi_{\sigma(g)}$, which is an inner automorphism. So in fact we do have $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$.

4.4.6 Prove that characteristic subgroups are normal. Give an example of a normal subgroup that is not characteristic.

Assume $H \mathrm{char} G$. By definition, this means that $\sigma(H) = H$ for every automorphism $\sigma \in \mathrm{Aut}(G)$. In particular, $\varphi_g(H) = H$ for every inner automorphism $\varphi_g \in \mathrm{Inn}(G)$, where $g \in G$ and $\varphi_g(x) = gxg^{-1}$ for every $x \in G$. But the equation $\varphi_g(H) = H$ simply asserts $gHg^{-1} = H$, that is, H is invariant under conjugation by every $g \in G$, so H is normal.

For every group G, there is an automorphism σ of the direct product $G \times G$ that interchanges the two factors, that is, $\sigma(g, g') = (g', g)$ for all $g, g' \in G$. This automorphism carries the normal subgroup $G \times 1 \triangleleft G \times G$ onto another normal subgroup $1 \times G$. Since $1 \times G \neq G \times 1$, this shows that $G \times 1$ is a normal subgroup that is not characteristic in $G \times G$.

4.4.8(a)(b) Let G be a group with subgroups H and K with $H \leq K$.

(a) Prove that if H is characteristic in K and K is normal in G then H is normal in G.

To show $H \triangleleft G$ we must show that H is invariant under any inner automorphism of G. Let $g \in G$ and let φ_g be conjugation by g, the associated inner automorphism of G associated with g. Since K is normal in G, we have $\varphi_g(K) = K$, so the restriction of φ_g to K is an automorphism of K. But H is characteristic in K, so it is left unchanged by this automorphism of K, that is, $\varphi_g(H) = H$, as desired.

(b) Prove that if H is characteristic in K and K is characteristic in G then H is characteristic in G. Use this to prove the Klein 4-group V_4 is characteristic in S_4.

Let σ be an automorphism of G. Then $\sigma(K) = K$ since K is characteristic in G. Therefore the restriction of σ to K is an automorphism of K. But H is characteristic in K, so it is left unchanged by this automorphism, hence $\sigma(H) = H$. Thus every automorphism of G fixes H, hence H is characteristic in G.

There are four copies of the Klein 4-group V_4 inside S_4, namely $K_1 := \{(), (12), (34), (12)(34)\}$, $K_2 := \{(), (13), (24), (13)(24)\}$, $K_3 := \{(), (14), (23), (14)(23)\}$, and $K_4 := \{(), (12)(34), (13)(24)\}$. By Prop. 17(4) of 4.4, all automorphisms of S_n are inner automorphisms whenever $n \neq 6$. The inner automorphism of S_n induced by a permutation $\sigma \in S_n$ (conjugation by σ) is the same as the automorphism induced on S_n by permuting the elements of $\{1, \ldots, n\}$ according to σ. There are automorphisms of S_4 (induced by permutations of the underlying set $\{1, 2, 3, 4\}$) that interchange K_1, K_2, and K_3. Thus none of K_1, K_2, and K_3 are characteristic in S_4. On the other hand, it is easy to check that every permutation of $\{1, 2, 3, 4\}$ in S_4 takes K_4 to K_4, so K_4 is characteristic in S_4.

All the elements of K_4 are even, so they lie in the subgroup A_4 of even permutations of S_4. Thus $K_4 \leq A_4 \leq S_4$. K_4 is the only subgroup of A_4 of order 4, by 3.5.9. Therefore K_4 is characteristic in A_4. We know A_4 is a normal subgroup of S_4 (because it has index 2), but A_4 is also characteristic in S_4 (proof?), so K_4 is characteristic in S_4.