Homework #8, due 10/21/09 = **4.1.1**, **4.1.2**, **4.1.10**, **4.2.8**

4.1.1 Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$ (G_a is the stabilizer of a). Deduce that if G acts transitively on A then the kernel of the action is $\bigcap_{g \in G} gG_ag^{-1}$.

Assume G acts on A, $a, b \in A$, $g \in G$, and $b = g \cdot a$. Let $k \in G_b$. Then $k \cdot b = b$, so

$$\begin{aligned}
(g^{-1}kg) \cdot a &= (g^{-1}k) \cdot (g \cdot a) \\
&= (g^{-1}k) \cdot b && g \cdot a = b \\
&= g^{-1} \cdot (k \cdot b) \\
&= g^{-1} \cdot b && k \cdot b = b \\
&= g^{-1} \cdot (g \cdot a) && g \cdot a = b \\
&= (g^{-1}g) \cdot a \\
&= 1 \cdot a \\
&= a
\end{aligned}$$

hence $g^{-1}kg \in G_a$. This proves $g^{-1}G_bg \subseteq G_a$, which implies $G_b \subseteq gG_ag^{-1}$.

For the opposite inclusion it is enough to assume $k \in G_a$ and show $gkg^{-1} \in G_b$. From $k \in G_a$ we have $k \cdot a = a$. Then

$$\begin{aligned}
(gkg^{-1}) \cdot b &= (gkg^{-1}) \cdot (g \cdot a) && b = g \cdot a \\
&= (gk) \cdot (g^{-1} \cdot (g \cdot a)) \\
&= (gk) \cdot ((g^{-1}g) \cdot a) \\
&= (gk) \cdot (1 \cdot a) \\
&= (gk) \cdot a \\
&= g \cdot (k \cdot a) \\
&= g \cdot a && k \cdot a = a \\
&= b && g \cdot a = b
\end{aligned}$$

so $gkg^{-1} \in G_b$, as desired.

In general, the kernel K of the action of a group G on a set A is the intersection of the stabilizers of all the elements of a, that is, $K = \bigcap_{b \in A} G_b$. Assume the action of G is transitive. Then there is just one orbit, namely $A = O_a$, so

$$K = \bigcap_{b \in A} G_b = \bigcap_{b \in O_a} G_b = \bigcap_{b \in \{g \cdot a | g \in G\}} G_b = \bigcap_{g \in G} G_{g \cdot a} = \bigcap_{g \in G} gG_ag^{-1}$$

4.1.2 Let G be a permutation group on the set A (i.e., $G \leq S_A$), let $\sigma \in G$ and let $a \in A$. Prove that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$.

To prove $\sigma G_a \sigma^{-1} \subseteq G_{\sigma(a)}$, it is enough to assume $g \in G_a$ and prove

$$(\sigma g \sigma^{-1})(\sigma(a)) = \sigma(a).$$

From $g \in G_a$ we have $g(a) = a$, so

$$(\sigma g \sigma^{-1})(\sigma(a)) = \sigma(g(\sigma^{-1}(\sigma(a)))) = \sigma(g((\sigma^{-1}\sigma)(a)) = \sigma(g(1(a)) = \sigma(g(a)) = \sigma(a),$$

as desired.

To prove $\sigma G_a \sigma^{-1} \supseteq G_{\sigma(a)}$, assume $g \in G_{\sigma(a)}$. Then $g(\sigma(a)) = \sigma(a)$, so $(\sigma^{-1} g \sigma)(a) = \sigma^{-1}(g(\sigma(a))) = \sigma^{-1}(\sigma(a)) = (\sigma^{-1}\sigma)(a) = 1(a) = a$. This shows $\sigma^{-1} g \sigma \in G_a$. It follows that $g \in \sigma G_a \sigma^{-1}$, which completes the proof that $G_{\sigma(a)} \subseteq \sigma G_a \sigma^{-1}$.

4.1.10 Let K and K be subgroups of the group G. For each $x \in G$ define the HK double coset of x in G to be the set $HxK = \{hxk | h \in H, k \in K\}$.

(a) Prove that HxK is the union of the left cosets $x_1 K, \ldots, x_n K$ where $\{x_1, \ldots, x_n K\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K.

To prove (a) it is enough to observe that $HxK = \bigcup_{h \in H} hxK = \bigcup\{h(xK) | h \in H\} = \bigcup O_{xK}$.

(b) Prove that HxK is the union of right cosets of H.

To prove (b) it is enough to observe that $HxK = \bigcup_{k \in K} Hxk = \bigcup\{(Hx)k | k \in K\}$.

(c) Show that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double cosets partitions G.

Every element $g \in G$ must belong to an HK double coset because $1 \in H$ and $1 \in K$, hence $g = 1g1 \in HgK$. Therefore the union of the HK double cosets is G. It remains to show that double cosets are either the same or disjoint. For this it is enough to assume that two double cosets have a non-empty intersection and show they are the same. So suppose $g \in HxK \cap HyK$. Then $g = hxk = h'yk'$ for some $h, h' inH$ and $k, k' inK$. Then $x = h^{-1}h'yk'k^{-1}$, but $h^{-1}h' \in H$ and $k'k^{-1} \in K$ since H and K are subgroups, so $x \in HyK$. But this implies $HxK \subseteq H(HyK)K = (HH)y(KK) = HyK$. Similarly, $HyK \subseteq HxK$, so $HxK = HyK$, as desired.

(d) Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$.

Let H act on the left cosets of K in $\{hxK | h \in H\}$ by left multiplication. Clearly there is a single orbit, namely $\{hxK | h \in H\}$, and this is the orbit of xK, so $O_{xK} = \{hxK | h \in H\}$. For any action, the size of an orbit is equal to the index (in the acting group) of the stabilizer of an element of that orbit. In this particular case, this fact tells us that $|O_{xK}| = |H : H_{xK}|$.

Next we calculate H_{xK}. Note that for any $h \in H$ we have $h \in H_{xK}$ iff $hxK = xK$ iff $x^{-1}hxK = K$ iff $x^{-1}hx \in K$ (since $K \leq G$) iff $h \in xKx^{-1}$. Therefore $H_{xK} = H \cap xKx^{-1}$. From this it follows by our previous equation that

$$|O_{xK}| = |H : H \cap xKx^{-1}|.$$

Now HxK is the union of the left cosets of K in $\{hxK | h \in H\}$, and each of these cosets has size $|K|$, so $|HxK|$ is equal to $|K|$ times the number of cosets in $\{hxK | h \in H\}$, which is $|O_{xK}| = |H : H \cap xKx^{-1}|$. Therefore

$$|HxK| = |K||H : H \cap xKx^{-1}|.$$

(e) Prove that $HxK = |H| \cdot |K : K \cap x^{-1}Hx|$.

This part could probably be easily deduced from part (d) by symmetry, but that might not explicitly show why x becomes x^{-1}, so I'm going to repeat the entire proof of part (d) with all the changes (such as interchanging H and K) needed to turn it into a proof of part (e). Here goes:

Let K act on the right cosets of H in $\{Hxk|k \in K\}$ by right multiplication by inverses (which means $k \cdot (Hxk') = Hxk'k^{-1}$ for all $k, k' \in K$). Clearly there is a single orbit, namely $\{Hxk|k \in K\}$, and this is the orbit of Hx, so $O_{Hx} = \{Hxk|k \in K\}$. For any action, the size of an orbit is equal to the index (in the acting group) of the stabilizer of an element of that orbit. In this particular case, this fact tells us that

$$|O_{Hx}| = |K : K_{Hx}|.$$

Next we calculate K_{Hx}. Note that for any $k \in K$ we have $k \in K_{Hx}$ iff $Hxk^{-1} = Hx$ iff $H = Hxkx^{-1}$ iff $xkx^{-1} \in H$ (since $H \leq G$) iff $k \in x^{-1}Hx$. Therefore $K_{Hx} = K \cap x^{-1}Hx$. From this it follows by our previous equation that

$$|O_{Hx}| = |K : K \cap x^{-1}Hx|.$$

Now HxK is the union of the right cosets of H in $\{Hxk|k \in K\}$, and each of these cosets has size $|H|$, so $|HxK|$ is equal to $|H|$ times the number of cosets in $\{Hxk|k \in K\}$, which is $|O_{Hx}| = |K : K \cap x^{-1}Hx|$. Therefore

$$|HxK| = |H||K : K \cap x^{-1}Hx|.$$

4.2.8 Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.

Let G act on the left cosets of H by left multiplication. Let K be the kernel of this action. By Theorem 3 of §4.2, K is a normal subgroup of G that is contained in H. The action induces a group homomorphism φ from G into the permutation group $S_{G/H}$ of the left cosets of H. The number of left cosets is n, so the size of the permutation group $S_{G/H}$ is $n!$. By the First Isomorphism Theorem, G/K is isomorphic to a subgroup of $S_{G/H}$, so the order of G/K must divide the order of $S_{G/H}$, which is $n!$. But the order G/K is $|G : K|$, so $|G : K|$ must divide $n!$. It follows that $|G : K| \leq n!$.