Homework #6, due 10/7/09 = **3.1.3**, **3.1.24**, **3.1.36**, **3.1.37**

3.1.3 Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian. Give an example of a non-abelian group G containing a proper normal subgroup N such that G/N is not abelian.

The conjugate of B by any element a of A is B again, that is,

$$\begin{aligned} aBa^{-1} &= \{aba^{-1}|b \in B\} \\ &= \{baa^{-1}|b \in B\} && A \text{ is abelian, so } ab = ba \\ &= \{b|b \in B\} = B \end{aligned}$$

This shows B is normal in A, so A/B is a group. This quotient group A/B is abelian since for all $a, a' \in A$ we have

$$\begin{aligned} aBa'B &= aa'BB && \text{since } B \text{ is normal, hence } Ba' = a'B \\ &= a'aBB && \text{since } A \text{ is abelian, } aa' = a'a \\ &= a'BaB && \text{since } B \text{ is normal, hence } aB = Ba \end{aligned}$$

Let G be the direct product $\mathbb{Z}_6 \times S_3$. Then G is not abelian because S_3 is not abelian. Let N be the subgroup $1 \times S_3$ of G. Then, by **3.1.37** below, N is a normal subgroup of G and the quotient group G/N is isomorphic to $\mathbb{Z}_6$, which is abelian.

3.1.24 Prove that if $N \lhd G$ and H is any subgroup of G then $N \cap H \lhd H$.

To show $N \cap H \lhd H$ we must show that the conjugate of $N \cap H$ by an element of h is $N \cap H$. So let $h \in H$. Then $h(N \cap H)h^{-1} = hNh^{-1} \cap hHh^{-1}$ but $hNh^{-1} = N$ since N is normal in G, and $hHh^{-1} = H$ since H is a subgroup of G, so $g(N \cap H)g^{-1} = N \cap H$, as desired.

3.1.36 Prove that if $G/Z(G)$ is cyclic then G is abelian.

Assume $G/Z(G)$ is cyclic with generator $xZ(G)$. Let $g_1, g_2 \in G$. Then $g_1 \in g_1Z(G) = (xZ(G))^{a_1} = x^{a_1}Z(G)$ for some $a_1 \in \mathbb{Z}$, so there is some $z_1 \in Z(G)$ such that $g_1 = x^{a_1}z_1$. Similarly, there are $a_2 \in \mathbb{Z}$ and $z_2 \in Z(G)$ such that $g_2 = x^{a_2}z_2$. Then

$$\begin{aligned} g_1g_2 &= x^{a_1}z_1x^{a_2}z_2 \\ &= x^{a_1}x^{a_2}z_1z_2 && \text{since } z_1 \in Z(G) \\ &= x^{a_1+a_2}z_2z_1 && \text{since } z_1 \in Z(G) \\ &= x^{a_2}x^{a_1}z_2z_1 \\ &= x^{a_2}z_2x^{a_1}z_1 && \text{since } z_2 \in Z(G) \end{aligned}$$

3.1.37 Let A and B be groups. Show that $\{(a,1)|a \in A\}$ is a normal subgroup of $A \times B$ and the quotient of $A \times B$ by this subgroup is isomorphic to B.

Let $A \times 1 = \{(a,1)|a \in A\}$. To show $A \times 1$ is normal in $A \times B$, we must show the conjugate of $A \times 1$ by an arbitrary element (a,b) of $A \times B$ is again $A \times 1$, that is,

$$\begin{aligned} (a,b)(A \times 1)(a,b)^{-1} &= (a,b)\{(a,1)|a \in A\}(a^{-1},b^{-1}) \\ &= \{(a,b)(a,1)(a^{-1},b^{-1})|a \in A\} \\ &= \{(aaa^{-1}, b1b^{-1})|a \in A\} \end{aligned}$$

$$= \{(a,1) | a \in A\}$$
$$= A \times 1$$

Define $\varphi : A \times B \to B$ by $\varphi(a,b) = b$. Then φ is a homomorphism since, for all $(a,b),(a',b') \in A \times B$, $\varphi((a,b)(a',b')) = \varphi(aa',bb') = bb' = \varphi(a,b)\varphi(a',b')$. The kernel of φ is $A \times 1$, because, for all $(a,b) \in A \times B$, if $\varphi(a,b) = 1$, then $b = 1$, hence $(a,b) = (a,1) \in A \times 1$. By the First Isomorphism Theorem, $(A \times B)/(A \times 1) \cong B$.