Homework #5, due 9/30/09 = **2.2.10**, **2.3.24**, **2.4.3**, **2.5.1**

2.2.10 Let H be a subgroup of order 2 in group G. Show that $N_G(H) = C_G(H)$. Deduce that if $N_G(H) = G$ then $H \leq Z(G)$.

If $x \in C_G(H)$ then $xh = hx$ for every $h \in H$. But $xh = hx$ implies $xhx^{-1} = h$, so $xhx^{-1} = h$ for every $h \in H$. This latter statement implies $xHx^{-1} = H$, so $x \in N_G(H)$. Thus we have $N_G(H) \supseteq C_G(H)$.

Now suppose $x \in N_G(H)$. Then $H = xHx^{-1}$. Since H has order 2, H has exactly 2 elements, one of which must be the identity element 1 of G. Let the other element of H be h, so that $H = \{1, h\}$. From $H = xHx^{-1}$ we get $\{1, h\} = x\{1, h\}x^{-1} = \{xx^{-1}, xhx^{-1}\} = \{1, xhx^{-1}\}$. This implies that either $1 = xhx^{-1}$ or $h = xhx^{-1}$. The former is impossible because if $1 = xhx^{-1}$ then $h = x^{-1}x = 1$, contradicting our assumption that H has two (distinct) elemnents 1 and h. Consequently $h = xhx^{-1}$, which implies $hx = xh$. Clearly x commutes with 1, and x commutes with h as well by this last equation, so x commutes with all (both) elements of H, and therefore $x \in C_G(H)$. This shows the opposite inclusion $N_G(H) \subseteq C_G(H)$, so we have $N_G(H) = C_G(H)$.

Suppose $N_G(H) = G$. From what we have proved so far it follows that $C_G(H) = G$, which says that every element of G commutes with every element of H. The center $Z(G)$ is the set of elements that commute with every element of G. But we just stated that every element of H is such an element, so H is included in the center of G, i.e., $H \subseteq Z(G)$. But both H and $Z(G)$ are subgroups of G, so this last inclusion implies $H \leq Z(G)$.

2.3.24 Let G be a finite group and let $x \in G$.

(a) Prove that if $g \in N_G(\langle x \rangle)$ then $gxg^{-1} = x^a$ for some $a \in \mathbb{Z}$.

From the assumption $g \in N_G(\langle x \rangle)$ we have $g\langle x \rangle g^{-1} = \langle x \rangle$. But $gxg^{-1} \in N_G(\langle x \rangle)$ so $gxg^{-1} \in \langle x \rangle = \{x^a : a \in \mathbb{Z}\}$ so there is some $a \in \mathbb{Z}$ such that $gxg^{-1} = x^a$.

(b) Prove conversely that if $gxg^{-1} = x^a$ for some $a \in \mathbb{Z}$ then $g \in N_G(\langle x \rangle)$.

Assume $gxg^{-1} = x^a$ and $a \in \mathbb{Z}$. Let $b \in \mathbb{Z}$. If $b > 0$ then

$$\begin{aligned} gx^b g^{-1} &= g \overbrace{xxx \cdots x}^{b} g^{-1} \\ &= \overbrace{gxg^{-1}gxg^{-1}gxg^{-1} \cdots gxg^{-1}}^{b} \\ &= (gxg^{-1})^b \\ &= (x^a)^b \\ &= x^{ab} \end{aligned}$$

and if $b < 0$ then

$$\begin{aligned} gx^b g^{-1} &= g \overbrace{x^{-1}x^{-1}x^{-1} \cdots x^{-1}}^{-b} g^{-1} \\ &= \overbrace{gx^{-1}g^{-1}gx^{-1}g^{-1}gx^{-1}g^{-1} \cdots gx^{-1}g^{-1}}^{-b} \\ &= (gx^{-1}g^{-1})^{-b} \end{aligned}$$

$$
\begin{aligned}
&= ((gxg^{-1})^{-1})^{-b} \\
&= (gxg^{-1})^{b} \\
&= (x^a)^b \\
&= x^{ab}
\end{aligned}
$$

Therefore $gx^bg^{-1} = x^{ab}$ for all $b \in \mathbb{Z}$, so

$$
\begin{aligned}
g\langle x\rangle g^{-1} &= g\{x^b : b \in \mathbb{Z}\}g^{-1} \\
&= \{gx^bg^{-1} : b \in \mathbb{Z}\} \\
&= \{x^{ab} : b \in \mathbb{Z}\} \\
&\subseteq \langle x\rangle
\end{aligned}
$$

But G is finite, so $\langle x\rangle$ is finite, and $g\langle x\rangle g^{-1}$ is a subgroup isomorphic to $\langle x\rangle$, so $g\langle x\rangle g^{-1}$ and $\langle x\rangle$ must have the same cardinality and hence are equal: $g\langle x\rangle g^{-1} = \langle x\rangle$, i.e., g is in the normalizer in G of $\langle x\rangle$.

2.4.3 Prove that if H is an abelian subgroup of G then $\langle H, Z(G)\rangle$ is abelian. Give an explicit example of an abelian subgroup H of a group G such that $\langle H, C_G(H)\rangle$ is not abelian.

First note that all the generators of $\langle H, Z(G)\rangle$ commute with each other: suppose $x, y \in H \cup Z(G)$. If x and y are both in H then $xy = yx$ since H is abelian. If either one of them (say x) is in the center $Z(G)$, then x commutes with *all* elements of G, including y. A proof by induction (not given here) shows that a subgroup generated by elements that commute with each other must be abelian. Thus $\langle H, Z(G)\rangle$ is abelian.

For an example, consider the direct product $H \times K$ of an abelian group H with a non-abelian group K. The subgroup $H \times 1 = \{(h, 1) : h \in H\}$ is abelian (by a simple calculation). Every element of the subgroup $1 \times K = \{(1, k) : k \in K\}$ commutes with every element of $H \times 1$, because $(1, k)(h, 1) = (h, k) = (h, 1)(1, k)$ for all $h \in H$ and all $k \in K$. This shows that $1 \times K$ is included in the centralizer $C_G(H \times 1)$ of $H \times 1$. But since $1 \times K$ is not abelian, it follows that neither the centralizer of H in G, nor any subgroup containing it, including $\langle H, C_G(H)\rangle$, is not abelian.

2.5.1 Let H and K be subgroups of G Exhibit all possible sublattices which show only G, 1, H and K and their joins and intersections. What distinguishes the different drawings?

There are four possible 2-chains: $1 = H = K < G$, $1 < H = K = G$, $1 = H < K = G$, $1 = K < H = G$; five possible 3-chains: $1 < H < K = G$, $1 = K < H < G$, $1 < H = K < G$, $1 < K < H = G$, $1 = H < K < G$; two possible 4-chains: $1 < H < K < G$, $1 < K < H < G$. The diamond is possible: $1 = H \cap K < H, K < \langle H \cup K\rangle = G$; the diamond with top is possible: $1 = H \cap K < H, K < \langle H \cup K\rangle < G$; the diamond with tail is possible: $1 < H \cap K < H, K < \langle H \cup K\rangle = G$. The six-element lattice consisting of a diamond with top and tail is possible: $1 < H \cap K < H, K < \langle H \cup K\rangle < G$. What distinguishes the different drawings? I have no idea what this question means. Some drawings are of the same lattice, but with different labelings. Perhaps that is what the authors want.