Homework #4, due 9/23/09 = **1.6.10**, **1.6.17**, **1.7.4**, **1.7.17**, **2.1.8**

1.6.10 Fill in the details in the proof that the symmetric groups S_Δ and S_Ω are isomorphic if $|\Delta| = |\Omega|$ as follows: let $\theta : \Delta \to \Omega$ be a bijection. Define $\varphi : S_\Delta \to S_\Omega$ by $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1}$ for all $\sigma \in S_\Delta$, and prove the following

(a) φ is well-defined, that is, if σ is a permutation of Δ then $\theta \circ \sigma \circ \theta^{-1}$ is a permutation of Ω.

Since θ is a bijection it has an inverse function $\theta^{-1} : \Omega \to \Delta$ such that $\theta \circ \theta^{-1}$ is the identity function I_Ω on Ω and $\theta^{-1} \circ \theta$ is the identity function I_Δ on Δ. Suppose σ is a permutation of Δ. This implies that $\sigma : \Delta \to \Delta$ so we have the following situation,

$$\Omega \xrightarrow{\theta} \Delta \xrightarrow{\sigma} \Delta \xrightarrow{\theta^{-1}} \Omega$$

and we may compose θ, σ, and θ^{-1} to obtain a function $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1} : \Omega \to \Omega$. Since σ is a permutation it has an inverse $\sigma^{-1} : \Delta \to \Delta$, so

$$\Omega \xrightarrow{\theta} \Delta \xrightarrow{\sigma^{-1}} \Delta \xrightarrow{\theta^{-1}} \Omega$$

Let $\tau = \theta \circ \sigma^{-1} \circ \theta^{-1}$. Then

$$\begin{aligned}
\varphi(\sigma) \circ \tau &= (\theta \circ \sigma \circ \theta^{-1}) \circ (\theta \circ \sigma^{-1} \circ \theta^{-1}) \\
&= \theta \circ \sigma \circ (\theta^{-1} \circ \theta) \circ \sigma^{-1} \circ \theta^{-1} \\
&= \theta \circ \sigma \circ I_\Delta \circ \sigma^{-1} \circ \theta^{-1} \\
&= \theta \circ (\sigma \circ \sigma^{-1}) \circ \theta^{-1} \\
&= \theta \circ I_\Delta \circ \theta^{-1} \\
&= \theta \circ \theta^{-1} \\
&= I_\Delta
\end{aligned}$$

and

$$\begin{aligned}
\tau \circ \varphi(\sigma) &= (\theta \circ \sigma^{-1} \circ \theta^{-1}) \circ (\theta \circ \sigma \circ \theta^{-1}) \\
&= \theta \circ \sigma^{-1} \circ (\theta^{-1} \circ \theta) \circ \sigma \circ \theta^{-1} \\
&= \theta \circ \sigma^{-1} \circ I_\Delta \circ \sigma \circ \theta^{-1} \\
&= \theta \circ (\sigma^{-1} \circ \sigma) \circ \theta^{-1} \\
&= \theta \circ I_\Delta \circ \theta^{-1} \\
&= \theta \circ \theta^{-1} \\
&= I_\Delta
\end{aligned}$$

Since $\varphi(\sigma)$ has τ as left and right inverse, it follows that $\varphi(\sigma)$ is a bijection by Prop 0.1.1(3). Since its domain and codomain are the same set Ω, this means that $\varphi(\sigma)$ is a permutation of Ω.

(b) φ is a bijection from S_Δ onto S_Ω [Find a 2-sided inverse for φ.]

Define a function ψ on S_Ω by $\psi(\xi) = \theta^{-1} \circ \xi \circ \theta$ for every ξ in S_Ω. Note that, since θ is a bijection, its inverse θ^{-1} is also a bijection, so the argmument above in part (a), with Δ and Ω interchanged and θ replaced with θ^{-1}, shows that $\psi(\xi)$ is a permutation of Δ. Thus ψ maps S_Ω into S_Δ. It turns out to be a 2-sided inverse of φ because, for every σ in S_Δ,

$$\begin{aligned}\psi(\varphi(\sigma)) &= \psi(\theta \circ \sigma \circ \theta^{-1}) \\ &= \theta^{-1} \circ (\theta \circ \sigma \circ \theta^{-1}) \circ \theta \\ &= (\theta^{-1} \circ \theta) \circ \sigma \circ (\theta^{-1} \circ \theta) \\ &= I_\Delta \circ \sigma \circ I_\Delta = \sigma\end{aligned}$$

and, for every ξ in S_Ω,

$$\begin{aligned}\varphi(\psi(\xi)) &= \varphi(\theta^{-1} \circ \xi \circ \theta) \\ &= \theta \circ (\theta^{-1} \circ \xi \circ \theta) \circ \theta^{-1} \\ &= (\theta \circ \theta^{-1}) \circ \xi \circ (\theta \circ \theta^{-1}) \\ &= I_\Omega \circ \xi \circ I_\Omega = \xi\end{aligned}$$

Since φ has a 2-sided inverse, it follows that φ is a bijection by Prop 0.1.1(3).

(c) φ is a homomorphism, that is, $\varphi(\sigma \circ \tau) = \varphi(\sigma) \circ \varphi(\tau)$ for all $\sigma\tau \in S_\Omega$. Then

$$\begin{aligned}\varphi(\sigma \circ \tau) &= \theta \circ (\sigma \circ \tau) \circ \theta^{-1} \\ &= \theta \circ \sigma \circ I_Delta \circ \tau \circ \theta^{-1} \\ &= \theta \circ \sigma \circ (\theta^{-1} \circ \theta) \circ \tau \circ \theta^{-1} \\ &= (\theta \circ \sigma \circ \theta^{-1}) \circ (\theta \circ \tau \circ \theta^{-1}) \\ &= \varphi(\sigma) \circ \varphi(\tau)\end{aligned}$$

so φ is indeed a homomorphism.

Parts (b) and (c) show that φ is a bijection and a homormorphism, i.e., it is an isomorphism.

1.6.17 Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

Assume $g \mapsto g^{-1}$ is a homomorphism. Then, for arbitrary elements $g, h \in G$, we have $(gh)^{-1} = g^{-1}h^{-1}$ by the homomorphism condition, but in every group we also have $(gh)^{-1} = h^{-1}g^{-1}$. From these two equations we get $g^{-1}h^{-1} = h^{-1}g^{-1}$. This last equation holds for all g and h, so we may apply it also to g^{-1} and h^{-1}, obtaining $(g^{-1})^{-1}(h^{-1})^{-1} = (h^{-1})^{-1}(g^{-1})^{-1}$, which simplifies to $gh = hg$. Thus any two elements of G commute, and G is therefore abelian.

Assume G is abelian. Then the homomorphism condition holds for the inversion map because, for all $g, h \in G$,

$$\begin{aligned}(gh)^{-1} &= h^{-1}g^{-1} \\ &= g^{-1}h^{-1} && G \text{ is abelian}\end{aligned}$$

1.7.4 Let G be a group acting on a set A and fix some $a \in A$. Show that the following sets are subgroups of G.

(a) The kernel $\{g \in G | \forall a \in A(g \cdot a = a)\}$ of the action.

First we prove that this set is closed under the group operation of G. Suppose g and h are in the kernel. Then, for every $a \in A$,

$$\begin{aligned}
(gh) \cdot a &= g \cdot (h \cdot a) && \text{action axiom} \\
&= g \cdot a && h \text{ is in the kernel} \\
&= a && g \text{ is in the kernel}
\end{aligned}$$

so gh is also in the kernel. Next we show the kernel is closed under inverses. Assume g is in the kernel and $a \in A$. Then $g \cdot a = a$. Applying g^{-1} to both sides of this last equation produces $g^{-1} \cdot (g \cdot a) = g^{-1} \cdot a$. By the first action axiom, $(g^{-1}g) \cdot a = g^{-1} \cdot a$, but $g^{-1}g = 1$ and the second action axiom says that $1 \cdot a = a$, so we get $a = g^{-1} \cdot a$. This shows g^{-1} is also in the kernel.

(b) The *stabilizer* $G_a = \{g \in G | g \cdot a = a\}$ of a in G.

For closure, assume g and h are in the stabilizer of a in G. Then $g \cdot a = a$ and $h \cdot a = a$, so, by the first action axiom, $(gh) \cdot a = g \cdot (h \cdot a) = g \cdot a = a$. Thus gh is also in the stabilizer of a. Furthermore, $g^{-1} \cdot (g \cdot a) = g^{-1} \cdot a$, but $g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1 \cdot a = a$ by both action axioms, so $a = g^{-1} \cdot a$, i.e., g^{-1} is in the kernel of the action. Finally, note that 1 is in the kernel by the second action axiom ($1 \cdot a = a$ for all $a \in A$).

1.7.17 Let G be a group and let G act on itself by left conjugation, so each $g \in G$ maps G to G by $x \mapsto gxg^{-1}$. For fixed $g \in G$, prove that conjugation by g is an isomorphism from G onto G itself (i.e. is an automorphism of G). Deduce that x and gxg^{-1} have the same order for all $x \in G$ and that for any subset $A \subseteq G$, $|A| = |gAg^{-1}|$.

Let $\varphi(x) = gxg^{-1}$ for all $x \in G$. Clearly φ maps G into G. First we will show φ is a homomorphism. The homorphism condition holds for φ because

$$\varphi(xy) = g(xy)g^{-1} = gx1yg^{-1} = gx(g^{-1}g)yg^{-1} = (gxg^{-1})(gyg^{-1}) = \varphi(x)\varphi(y).$$

Next, φ is injective, for if $\varphi(x) = \varphi(y)$ then $gxg^{-1} = gyg^{-1}$, so by multiplying this equation of the left of g^{-1} and on the right by g we get $g^{-1}(gxg^{-1})g = g^{-1}(gyg^{-1})g$, hence $(g^{-1}g)x(g^{-1}g) = (g^{-1}g)y(g^{-1}g)$, hence $1x1 = 1y1$, hence $x = y$. Finally, φ is surjective because if $x \in G$ then $\varphi(y) = x$ where $y = g^{-1}xg$, since $\varphi(y) = \varphi(g^{-1}xg) = g(g^{-1}xg)g^{-1} = x$.

Isomorphisms preserve order, so $|x| = |\varphi(x)| = |gxg^{-1}|$ for all $x \in G$. An isomorphism is a bijection, and bijections preserve cardinality of sets, so for any subset $A \subseteq G$, $|A| = |gAg^{-1}|$.

2.1.8 Let H and K be subgroups of G. Prove that $H \cup K$ is a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$.

First the easy direction: assume that either $H \subseteq K$ or $K \subseteq H$. If the former, then $H \cup K = H$ so $H \cup K$ is a subgroup of G simply because H is already a subgroup of G. Similarly, if $K \subseteq H$ then $H \cup K = K$ and again $H \cup K$ is a subgroup of G.

Now for the interesting direction. Assume that $H \cup K$ is a subgroup of G. It will suffice to assume that neither $H \subseteq K$ nor $K \subseteq H$ and derive a contradiction. Since it is not the case that $K \subseteq H$, there must be some element $k \in K$ such that $k \notin H$. Similarly, since $H \nsubseteq K$, there must be some element $h \in H$ such that $h \notin K$.

Consider the element hk. We have $h \in H \subseteq H \cup K$ and $k \in K \subseteq H \cup K$, but $H \cup K$ is assumed to be a subgroup, so $H \cup K$ is closed under the group operation of G. Therefore $hk \in H \cup K$. This implies that either $hk \in H$ or $hk \in K$, but we'll get a contradiction in either case. Note that since H and K are subgroups, we have $h^{-1} \in H$ and $k^{-1} \in K$. If $hk \in H$ then $k = h^{-1}(hk) \in H$, contradicting $k \notin K$, while if $hk \in K$ then $(hk)k^{-1} \in K$, contradicting $h \notin K$.