ON ATKIN AND SWINNERTON-DYER CONGRUENCE RELATIONS (3)

LING LONG

Abstract. In the previous two papers with the same title ([LLY05] by W.C. Li, L. Long, Z. Yang and [ALL05] by A.O.L. Atkin, W.C. Li, L. Long), the authors have studied special families of cuspforms for noncongruence arithmetic subgroups. It was found that the Fourier coefficients of these modular forms at infinity satisfy three-term Atkin and Swinnerton-Dyer congruence relations which are the $p$-adic analogue of the three-term recursions satisfied by the coefficients of classical Hecke eigenforms.

In this paper, we first consider Atkin and Swinnerton-Dyer type congruences which generalize the three-term congruences above. These weaker congruences are satisfied by cuspforms for special noncongruence arithmetic subgroups. Then we will exhibit an infinite family of noncongruence cuspforms, each of which satisfies three-term Atkin and Swinnerton-Dyer type congruences for almost every prime $p$. Finally, we will study a particular space of noncongruence cuspforms. We will show that the attached $l$-adic Scholl representation is isomorphic to the $l$-adic representation attached to a classical automorphic form. Moreover, for each of the four residue classes of odd primes modulo 12 there is a basis so that the Fourier coefficients of each basis element satisfy three-term Atkin and Swinnerton-Dyer congruences in the stronger original sense.

1. Introduction

This paper is a continuation of two previous papers with the same title: [LLY05] by W.C. Li, L. Long, Z. Yang and [ALL05] by A.O.L. Atkin, W.C. Li, L. Long. Here, we continue to explore modular forms for noncongruence arithmetic subgroups. The serious study of these functions was initiated in the late 1960's by Atkin and Swinnerton-Dyer [ASD71] and further developed by Scholl [Sch85, Sch86, Sch88, etc]. In particular, under a general assumption Scholl has established a system of compatible $l$-adic representations of the absolute Galois group attached to each space of noncongruence cuspforms. In [LLY05, ALL05], two intricate cases have been exhibited in which noncongruence cuspforms are related closely to classical congruence automorphic forms as follows: From the $p$-adic point of view, in each of these cases there is a simultaneous (or semi-simultaneous) basis for almost...
all primes $p$ such that each basis function is an “eigenform” for the “$p$-
adic” Hecke operators $T_p$ (c.f. Section 2.6). Moreover, the “traces” of these $T_p$ are Hecke eigenvalues (up to at most an ideal class character). From
the representation point of view, in each case the associated $l$-adic Scholl
representation $\rho_l$ can be decomposed into a direct sum of 2-dimensional
subrepresentations, when $\rho_l$ is restricted to a suitable Galois subgroup. Furthermore, each subrepresentation is shown to be isomorphic to an $l$-adic
representation attached to a classical congruence automorphic form. In this
paper, we intend to give a more general discussion on the first aspect and a
further discussion on the second one.

In this paper, we continue to consider modular forms for noncongruence
character groups (following the notation of A.O.L. Atkin). To be precise,
a character group is an arithmetic subgroup $\Gamma$ which is normal in its con-
gruence closure $\Gamma^0$ (the smallest congruence subgroup which contains $\Gamma$)
with abelian quotient. For example, the lattice groups studied by Rankin
[Ran67] and all groups in [LLY05, ALL05, ?] are character groups and most
of them are noncongruence. The construction of these groups endows the
corresponding modular forms with special arithmetic properties.

This paper is organized in the following way: Section 2 is devoted to the
properties of modular forms for character groups. In Section 3, we study
the arithmetic of weight 3 cuspforms for a special family of noncongruence
character groups denoted by $\Gamma_n$. In particular, we will prove the following
theorem

**Theorem 1.** For every positive integer $n$ and almost every prime $p$, the
space $S_3(\Gamma_n)$ of weight 3 cuspforms for $\Gamma_n$ has a rational basis independent
of $p$ such that for every basis element $\sum_{n \geq 1} a(n) w^n$, there exist a natural
number $r$ depending on $p$ and $n$ and two algebraic numbers $A(p)$ and $B(p)$
with $|A(p)| \leq \left(\frac{2r}{r}\right)^r$ and $|B(p)| \leq p^{2r}$ such that for any positive integer $n$

$$
a(np^r) - A(p)a(n) + B(p)a(n/p^r) \equiv 0 \pmod{(pn)^2}$$

is integral at $p$.

The congruence relation (1) is weaker but more general than those Atkin
and Swinnerton-Dyer relations obtained in [LLY05, ALL05].

In Section 4, we investigate the arithmetic properties of weight 3 cusp-
forms for $\Gamma_6$ in the above family. This case is very similar to the one obtained
in [ALL05]. The main result is

**Theorem 2.** Let $\rho_{3,l,6} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(W_{3,l,6})$ be the $l$-adic Scholl repre-
sentations attached to weight 3 cuspforms for $\Gamma_6$. Then $\rho_{3,l,6}$ are isomorphic
up to semisimplification to the $l$-adic representations attached to a classical
congruence automorphic form. Moreover, for every prime $p \nmid 6l$, the space of weight 3 cuspforms has a basis, depending on the congruence class of $p$
modulo 12, such that for each basis element $\sum_{n \geq 1} a(n)w_n$ and all $n \in \mathbb{N}$

$$\frac{a(np) - A(p)a(n) + B(p)a(n/p)}{(pn)^2}$$

is integral at some place in $\mathbb{Z}[i]$ above $p$ where $A(p)$ and $B(p)$ are two algebraic numbers with $|A(p)| \leq 2p$ and $|B(p)| \leq p^2$. Moreover, $A(p)$ and $B(p)$ can be obtained from the coefficients of a congruence cuspform.

In this paper, we use $\Gamma$ to denote a finite index subgroup of $SL_2(\mathbb{Z})$ such that $-I_2 \notin \Gamma$ and use $\Gamma^0$ to denote its congruence closure. Let $X_\Gamma$ denote the compact modular curve for $\Gamma$. We assume all modular curves considered here are defined over $\mathbb{Q}$. Let $\mathcal{M}_\Gamma$ denote the field of meromorphic modular functions for $\Gamma$, i.e. the field of rational functions on $X_\Gamma$. In particular, $\mathcal{M}_\Gamma$ is a finite extension of $\mathcal{M}_{\Gamma^0}$. Let $k$ be a positive integer. By $S_k(\Gamma)$ we mean the space of weight $k$ holomorphic cuspforms for $\Gamma$. In the sequel, let $\omega_n = e^{2\pi i/n}$ and $L_n = \mathbb{Q}(\omega_n)$. For any field $K$, let $G_K = \text{Gal}(\overline{K}/K)$. Unless otherwise mentioned, we follow all other notation used in [ALL05].

2. Modular forms for character groups

2.1. Character groups.

Definition 1. An arithmetic subgroup $\Gamma$ is called a character group of another arithmetic subgroup $\Gamma^0$ if $\Gamma$ is normal in $\Gamma^0$ with abelian quotient.

Since the quotient group $\Gamma^0/\Gamma$ acts on the $\mathbb{C}$-vector space $S_k(\Gamma)$ by the stroke operator $\vert$, by the representation of finite abelian group, $S_k(\Gamma)$ can be decomposed into a direct sum of representation subspaces parameterized by the characters of $\Gamma^0/\Gamma$. Where $\chi(\zeta) = \omega_n$ and $\chi^j (1 \leq j \leq n)$ are the characters of $\Gamma^0/\Gamma$. Here $S_k(\Gamma^0, \chi^j)$ consists of functions $f$ in $S_k(\Gamma)$ such that $f|_\zeta = \omega_n^j f$. Thus, when $(j, n) > 1$, $f \in S_k(\Gamma^0, \chi^j)$ is a modular form for an intermediate group sitting between $\Gamma$ and $\Gamma^0$. So, $\bigoplus_{(j, n) > 1} S_k(\Gamma^0, \chi^j)$ consists of forms for supergroups of $\Gamma$; while $\bigoplus_{(j, n) = 1} S_k(\Gamma^0, \chi^j)$ consists of cuspforms genuinely belonging to $\Gamma$. Accordingly, we denote these two spaces by $S_k(\Gamma)^{\text{old}}$ and $S_k(\Gamma)^{\text{new}}$ respectively. Under the Petersson inner product, we have

$$S_k(\Gamma) = S_k(\Gamma)^{\text{new}} \bigoplus S_k(\Gamma)^{\text{old}}.$$
2.3. Special symmetry. Since \( \zeta \) normalizes \( \Gamma \), it induces an order \( n \) cyclic covering map from the modular curve \( X_\Gamma \) of \( \Gamma \) to the modular curve \( X_{\Gamma^0} \) of \( \Gamma^0 \). We use \( \zeta \) again to denote such an involution and assume its minimal field of definition to be \( \mathbb{Q}(\omega_n) \). Moreover, we assume that as a simple finite extension of \( \mathbb{M}_{\Gamma^0} \), \( \mathbb{M}_\Gamma \) is generated by \( t_n \) with minimal polynomial \( (t_n)^n - t \) for some \( t \in \mathbb{M}_{\Gamma^0} \) with rational Fourier coefficients at infinity. Then \( X_\Gamma \) satisfies a symmetry

\[ \zeta : t_n \mapsto \omega_n^{-1} t_n. \]  

Such a map \( \zeta \) is defined over \( L_n = \mathbb{Q}(\omega_n) \).

2.4. \( l \)-adic Scholl representations. Assume \( \dim_{\mathbb{C}} S_k(\Gamma) = d \). For any prime number \( l \), let \( \rho_{k,l} : G_\mathbb{Q} \to \text{Aut}(W_{k,l}) \) be the \( l \)-adic Scholl representation attached to \( S_k(\Gamma) \) which is unramified outside of a few primes \( \text{[Sch85]} \). Here \( W_{k,l} \) is a \( 2d \)-dimensional \( \mathbb{Q}_l \)-vector space. Denoted by \( N \) the product of all ramifying primes of \( \rho_{k,l} \) except \( l \) together with all prime divisors of \( n \). Let \( p \nmid Nl \) be a prime and \( F_p \) be the canonical Frobenius conjugacy class in the quotient of the decomposition group at \( p \) by the inertia group at \( p \). Scholl has also shown that the characteristic polynomial of \( \rho_{k,l}(F_p) \) for any \( p \nmid Nl \) has integral coefficients and the eigenvalues have the same absolute value \( p^{(k-1)/2} \).

When \( \Gamma \) is a character group as above. The map \( \zeta \) (3) endows \( W_{k,l} \otimes \mathbb{Q}_l(\omega_n) \) with an order \( n \) involution. Under our assumptions, \( W_{k,l} \otimes \mathbb{Q}_l(\omega_n) \) decomposes into eigenspaces of \( \zeta \). Let \( W_j \) be such an eigenspace with eigenvalue \( \omega_j^\delta \). Let

\[ W_{k,j}^{\text{new}} = \bigoplus_{(j,n)=1} W_j \quad \text{and} \quad W_{k,j}^{\text{old}} = \bigoplus_{(j,n)>1} W_j. \]

The space \( W_{k,j}^{\text{new}} \) is simply denoted by \( W_j^{\text{new}} \) when there is no ambiguity.

Lemma 3. Assume \( p \nmid Nl \) is a prime. In that case

\[ F_p \zeta = \zeta^p F_p, \quad \text{and} \quad \zeta F_p = F_p \zeta^\delta, \]

where \( \delta \) denotes the inverse of \( p \) in \( (\mathbb{Z}/n\mathbb{Z})^\times \).

Proof. Modulo the prime ideal \( (p) \), we have

\[ F_p \zeta(t_n) = F_p(\omega_n^{-1} t_n) = \omega_n^{-p} t_n = \zeta^p F_p(t_n). \]
\[ \zeta F_p(t_n) = \omega_n^{-1} t_n = (\omega_n^{-p} t_n)^p = F_p \zeta^\delta(t_n). \]

\( \square \)

Consequently, for any \( w \in W_j \), \( \zeta(F_p w) = F_p \zeta^\delta w = \omega_n^\delta (F_p w) \). Thus \( F_p W_j \subseteq W_{j\delta} \). By iteration, \( F_p \) permutes \( W_j \) and

\[ F_p W_j = W_{j\delta}. \]
Since the group generated by all $F_p$ with $p \nmid N_l$ acts on $\{W_j \}_{j \in (\mathbb{Z}/n\mathbb{Z})^\times}$ transitively,
\[
\dim_{\mathbb{Q}(\omega_n)} W_j = \delta_p, \tag{4}
\]
where $\delta_p$ is an integer independent of $j$ in $(\mathbb{Z}/n\mathbb{Z})^\times$. Therefore
\[
\dim_{\mathbb{Q}(\omega_n)} W_{\text{new}} = \delta_p \cdot \phi(n), \tag{5}
\]
where $\phi(n)$ is the Euler number of $n$. For any $p \nmid n$, use $O_n(p)$ to denote the order of $p$ in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. Let $r = O_n(p)$. Moreover, we have
\[
\mathcal{L}_{j,p} = \bigoplus_{m=1}^{r} W_{j \hat{p}^m} \tag{6}
\]
is invariant under $F_p$.

**Corollary 4.** The spaces $W_{k,l}^{\text{new}}$ and $W_{k,l}^{\text{old}}$ are invariant under each $F_p$ when $p \nmid N_l$. Therefore as a Gal($\mathbb{Q}$/\mathbb{Q}$)-module, $W_{k,l} = W_{k,l}^{\text{new}} \bigoplus W_{k,l}^{\text{old}}$.

**Proof.** For any $p \nmid N_l$, $(\hat{p}, n) = 1$, and $(j, n) = (j\hat{p}, n)$. The assertions follow naturally. \qed

In the remaining part of this section, we will focus on $W_{k,l}^{\text{new}}$ and let $\rho_{k,l}^{\text{new}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(W_{k,l}^{\text{new}})$.

As we have extended the scalar field to include $\omega_n$, there exists a basis $\mathcal{B}$ of $W_{\text{new}}$ under which the matrix of $\zeta$ on $W_{\text{new}}$ is a diagonal matrix of the following block form
\[
\zeta = \begin{pmatrix}
\omega_n^{j_1} I_{\delta_p} & 0 & 0 & \cdots & 0 & 0 \\
0 & \omega_n^{j_2} I_{\delta_p} & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega_n^{j_{\phi(n)-1}} I_{\delta_p} & 0 \\
0 & 0 & 0 & \cdots & 0 & \omega_n^{j_{\phi(n)}} I_{\delta_p}
\end{pmatrix}, \tag{7}
\]
where $j_i$ runs through the set $(\mathbb{Z}/n\mathbb{Z})^\times$.

If $B$ is an operator on a vector space $L$, let Char($L, B$)($T$) denote the characteristic polynomial of $B$ on $L$ with variable $T$.

**Lemma 5.** Let $p \nmid N_l$ be a prime number and $j \in (\mathbb{Z}/n\mathbb{Z})^\times$. Let $r = O_n(p)$. Then
\[
\text{Char}(\mathcal{L}_{j,p}, F_p)(T) \in \mathbb{Q}(\omega_n)[T^r].
\]

**Proof.** Assume that under $\mathcal{B}$ the matrix of $F_p$ on $W^{\text{new}}$ is $(E_{i,j})_{1 \leq i,j \leq \phi(n)}$ where $E_{i,j}$ are $\delta_p \times \delta_p$ matrices. The commutativity $F_p \zeta = \zeta^p F_p$ in Lemma 3 and (7) imply that $E_{i,j} = (0)_{\delta_p \times \delta_p}$ are all-zero matrices unless $i \cdot (-p) + j = 0 \mod n$. So the matrix of $(F_p)^n$ restricted to $\mathcal{L}_{j,p}$ consists of block forms with diagonal blocks $(0)_{\delta_p \times \delta_p}$ unless $r \mid n$. Therefore the trace of $(F_p)^n$ on $\mathcal{L}_{j,p}$ is 0 unless $r \mid n$. Consequently the characteristic polynomial of $F_p$ on $\mathcal{L}_{j,p}$ is in terms of $T^r$. \qed
Lemma 6. Char\((L_{j,p}, F_p)(T) = Char(W_j, F_p^r)(T^r)\).

Proof. By Lemma 5, if \(\alpha\) is a solution of \(Char(L_{j,p}, F_p)^r(T) = 0\), so is \(\omega_r \alpha\). Hence, all the roots of \(Char(L_{j,p}, F_p)^r(T)\) are \(\{\omega_r \alpha_m\}_{1 \leq j \leq r, 1 \leq m \leq \delta_p}\).

On the other hand \(Char(W_j, F_p^r)(T)\) coincides with the characteristic polynomial of the \(\delta_p \times \delta_p\) matrix \(\prod_{m=1}^\delta E_{p-m_j,p-m+j}\). Since the order of the product does not effect the characteristic polynomial,

\[
Char(L_{j,p}, F_p^r)(T) = (Char(W_j, F_p^r)(T))^r = \prod_{1 \leq m \leq \delta_p} (T - \alpha_m^r)^r.
\]

Hence \(Char(W_j, F_p^r)(T) = \prod_{1 \leq m \leq \delta_p} (T - \alpha_m^r)\). It follows

\[
Char(W_j, F_p^r)(T^r) = \prod_{1 \leq m \leq \delta_p} (T^r - \alpha_m^r) = Char(L_{j,p}, F_p)(T).
\]

□

Proposition 7. The polynomial \(Char(W_{new}^r, F_p)(T)\) is in \(\mathbb{Z}[T^r]\) with all roots having the same absolute value \(p^{(k-1)/2}\).

Proof. By Lemma 5 and Scholl’s Theorem which says \(Char(W_{k,l}, F_p)(T) \in \mathbb{Z}[T]\) and its eigenvalues have the same absolute value \(p^{(k-1)/2}\), it suffices to show \(Char(W_{new}^r, F_p)(T) \in \mathbb{Z}[T]\). Let \(\Gamma'\) be an intermediate group \(\Gamma \subset \Gamma' \subset \Gamma^0\). By our assumption, \(X_{\Gamma'}\) has a model over \(\mathbb{Q}\). Let \(W_{k,l}(\Gamma')\) be the \(l\)-adic Scholl representation space associated with \(S_k(\Gamma')\). Then \(Char(W_{k,l}(\Gamma'), F_p)(T) \in \mathbb{Z}[T]\). When \(m = [\Gamma^0 : \Gamma']\) is a prime,

\[
Char(W_{k,l}^{\text{old}}(\Gamma'), F_p)(T) = Char(W_{k,l}(\Gamma^0), F_p)(T) \in \mathbb{Z}[T]
\]

and hence

\[
Char(W_{k,l}^{\text{new}}(\Gamma'), F_p)(T) \in \mathbb{Z}[T].
\]

By induction, for any divisor \(m|n\), \(Char(\bigoplus_{j,(j,n)=n/m} W_j, F_p)(T) \in \mathbb{Z}[T]\). In particular,

\[
Char(W_{k,l}^{\text{new}}, F_p)(T) = Char(\bigoplus_{j,(j,n)=n/n} W_j, F_p)(T) \in \mathbb{Z}[T^r].
\]

□

2.5. Induced representations. As a consequence of the above discussions we have for any \(p \nmid Nl\)

\[
\text{Tr}(\rho_{k,l}^{\text{new}}(F_p)) = 0\quad \text{if}\quad p \neq 1 \text{ mod } n.
\]

Since the images of all Frobenius elements will determine any Galois representation up to semisimplification, this implies that the character of \(\rho_{k,l}^{\text{new}}\) is invariant under any twisting by Dirichlet characters of modulus divisible by \(n\). Let \(W_j\) as above, \(L_n = \mathbb{Q}(\omega_n)\), and \(\rho_j : G_{L_n} \to \text{Aut}(W_j)\) for any \(j \in (\mathbb{Z}/n\mathbb{Z})^\times\). The group \(G_{L_n}\) is an index \(\phi(n)\) subgroup of \(G_{\mathbb{Q}}\) and \(W_j\) is
a $\delta_p$-dimensional representation space of $G_L$. Applying a result in [Ser77, Prop. 19], we obtain the following result.

**Proposition 8.** Let $j \in (\mathbb{Z}/n\mathbb{Z})^\times$. We have

$$\rho_{k,l}^{\text{new}} = \text{Ind}_{G_{Q(\omega_n)}}^{G_Q} \rho_j.$$

2.6. $p$-adic spaces and Atkin and Swinnerton-Dyer type congruences. Let $S_k(\Gamma, \mathbb{Z}_p) = S_k(\Gamma) \otimes \mathbb{Z}_p$, and $V$ be the $p$-adic Scholl space attached to $S_k(\Gamma)$ with $S_k(\Gamma, \mathbb{Z}_p)$ as a subspace [Sch85]. These $p$-adic spaces are endowed with the action of the Frobenius morphism, which is denoted by $F$. The operator $\zeta$ also acts on $V$ and its order is still $n$. Like before, we can define $V_j$ to be the $\omega_j$-eigenspace of $\zeta$ on $V$ (over $\mathbb{Q}_p(\omega_n)$) as well as $V^{\text{new}}$ and $V^{\text{old}}$ similarly.

In [ASD71], Atkin and Swinnerton-Dyer observed that some noncongruence modular forms satisfy three-term Hecke-like recursions in a $p$-adic sense. Their observations have been verified by Cartier for weight 2 cases [Car71] and Scholl [Sch85] for all 1-dimensional cases. Here, we will first define Atkin and Swinnerton-Dyer type congruence relations.

**Definition 2.** Let $K$ be a number field and $f = \sum_{n\geq 1} a(n)w^n$ a weight $k$ modular form. The function $f$ is said to satisfy the Atkin and Swinnerton-Dyer type congruence relation at $p$ given by polynomial $T^{2d} + A_1T^{2d-1} + \cdots + A_{2d} \in K[T]$ if for all $n \in \mathbb{Z}$,

$$\frac{a(np^d) + A_1a(np^{d-1}) + \cdots + A_da(n) + \cdots + A_{2d}a(n/p^d)}{(np)^{k-1}}$$

is integral at some place of $K$ above the prime number $p$.

To examine three-term Atkin and Swinnerton-Dyer congruence relations computationally, Atkin uses “$p$-adic” Hecke operators which we will explain. For any prime $p \nmid N$, define the $p$-adic Hecke operator to be $T_p = U_p + s(p) \cdot p^{k-1}V_p$ which acts on a weight $k$ form $f = \sum a(n)w^n$ as follows:

$$f|_{T_p} = \sum_n (a(np) + s(p)p^{k-1}a(n/p))w^n.$$

Here, $s(p)$ is an algebraic number with absolute value 1. It is called a “$p$-adic” Hecke operator as the $n$th coefficient of $f|_{T_p}$ is only determined up to modulo some suitable power of $p$ depending on $n$. In general, it is totally nontrivial to guess the right $s(p)$ values so that $T_p$ has “eigen” forms in $S_k(\Gamma)$. Moreover $s(p)$ are rarely Dirichlet characters. Unlike the classical Hecke operators which can be diagonalized simultaneously, these operators are a priori defined over various $p$-adic fields. It is clearly exceptional that the space $S_k(\Gamma)$ has a simultaneous eigen basis for all “$p$-adic” Hecke operators $T_p$, which is the case in [LLY05].
2.7. Atkin and Swinnerton-Dyer type congruences satisfied by modular forms for noncongruence character groups. Applying Theorem 5.6 in [Sch85], we have

**Theorem 9.** [Scholl, [Sch85]] For any prime \( p \nmid N \).

\[
\text{Char}(V, F)(T) = \text{Char}(W, F_p)(T) \in \mathbb{Z}[T].
\]

**Proposition 10** (Gang et al [FHL+05]). Let \( G \) be a finite group of automorphisms of the elliptic modular surface \( \pi : \mathcal{E}_\Gamma \rightarrow X_\Gamma \) associated with \( \Gamma \). Let \( \chi \) be an irreducible character of \( G \) and if \( V \) is a representation of \( G \), let \( V^\chi \) denote the \( \chi \)-isotypical subspace of \( V \). Let \( K/\mathbb{Q} \) be the field of definition of the representation whose character is \( \chi \) and let \( \lambda \) be a place of \( K \) above \( l \) and \( \wp \) be a place of \( K \) above \( p \). Finally, we let \( r \) to be the smallest positive integer such that \( (F_p)^r \in G_K \). Then

\[
\text{Char}((V \otimes K_\wp)^\chi, F^r)(T) = \text{Char}((W \otimes K_\lambda)^\chi, (F_p)^r)(T). \quad (8)
\]

In our case, we use \( G = \Gamma^0/\Gamma \) which is generated by \( \zeta \Gamma \). Then \( (V \otimes K_\wp)^\chi = V_j \) and \( (W \otimes K_\lambda)^\chi = W_j \).

**Corollary 11.** If \( j \in (\mathbb{Z}/n\mathbb{Z})^\times \), then \( \dim_{\mathbb{Q}_p(\omega_n)} V_j = \delta_p \) where \( \delta_p \) is even.

**Corollary 12.**

\[
\text{Char}(V_{\text{new}}, F)(T) = \text{Char}(W_{\text{new}}, F_p)(T) \in \mathbb{Z}[T].
\]

Moreover, the roots of these monic polynomials have the same absolute value \( p^{(k-1)/2} \).

**Theorem 13.** Assume \( \Gamma \) is a character group of \( \Gamma^0 \) with \( \Gamma^0/\Gamma = < \zeta \Gamma > \cong \mathbb{Z}_n \), \( X_\Gamma \) has a model defined over \( \mathbb{Q} \), and the action of \( \zeta \) on \( X_\Gamma \) is defined over \( \mathbb{Q}(\omega_n) \). Let \( p \nmid N \) be a prime, \( j \in (\mathbb{Z}/n\mathbb{Z})^\times \), and \( r = O_n(p) \). Assume the space \( V_j \) has a basis consisting of forms with Fourier coefficients in \( L_n \). Let \( H_j(T) = \text{Char}(L_j, F_p)(T) \in L_n[T^r] \). For every \( f \in V_j \) with coefficients in \( L_n \), \( f \) satisfies an Atkin and Swinnerton-Dyer type congruence at \( p \) given by \( H_j(T) \).

**Proof.** It follows from Scholl’s arguments in [Sch85].

In particular, when \( \delta_p = 2 \), we have

\[
H_j(T) = T^{2r} - A_j(p)T^r + B_j(p), \quad (9)
\]

where \( |A_j(p)| \leq \binom{2r}{r} p^{(k-1)/2} \), \( |B_j(p)| = p^{(k-1)}r \). The corresponding three-term recursion is weaker than the original three-term Atkin and Swinnerton-Dyer congruence relation when \( r > 1 \).
3. Atkin and Swinnerton-Dyer Type Congruences Satisfied by Cusps for $\Gamma_n$

In this section, we will use the results obtained in the previous section to derive three-term Atkin and Swinnerton-Dyer type congruence relations satisfied by weight 3 cusps for $\Gamma_n$.

3.1. The groups $\Gamma_n$. In [LLY05, ALL05], the following family of noncongruence character group is considered.

Let $\Gamma^1(5) = \{ \gamma \in SL_2(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix} \mod 5 \}.$

It is an index 12 congruence subgroup of $SL_2(\mathbb{Z})$ with 4 cusps $\infty, -2, 0, -5/2$. This group has four generators $\gamma_{\infty}, \gamma_{-2}, \gamma_0, \gamma_{-5/2}$ (each stabilizes one cusp as indicated by the subscripts) subject to one relation $\gamma_{\infty}\gamma_{-2}\gamma_0\gamma_{-5/2} = I_2$.

Let $\varphi_n$ be the homomorphism $\Gamma^1(5) \rightarrow \mathbb{C}^\times$

$$\gamma_{\infty} \mapsto \omega_n$$

$$\gamma_{-2} \mapsto \omega_n^{-1}$$

$$\gamma_0, \gamma_{-5/2} \mapsto 1.$$  

The kernel $\Gamma_n$ of $\varphi_n$ is an index $n$ noncongruence character group of $\Gamma^1(5)$. When $n \neq 1, 5$, $\Gamma_n$ is a noncongruence subgroup.

Let $E_1, E_2$ be the weight 3 Eisenstein series for $\Gamma^1(5)$ as in [LLY05]. A Hauptmodul for $\Gamma^1(5)$ is $t = E_1/E_2$ which generates $\mathfrak{M}_{\Gamma^1(5)}$ and $t_n = \sqrt{E_1/E_2}$ is a Hauptmodul for $\Gamma_n$. Such a group $\Gamma_n$ has two distinguished normalizers in $SL_2(\mathbb{Z})$: $\zeta = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \ A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$. In particular

$$E_1|A = E_2, \ E_2|A = -E_1, \ t|A = -\frac{1}{t}. \quad (10)$$

We use the following algebraic model for the elliptic modular surface $E_{\Gamma_n}$ associated to the group $\Gamma_n$ to investigate the roles of the above operators.

$$E_{\Gamma_n}: \quad y^2 = t_n^4(x^3 - \frac{1 + 12(t_n^3 - t_{5n}^3) + 14t_{2n}^2 + t_{4n}^4}{48t_{2n}^2}x$$

$$+ \frac{1 + 18(t_n^5 - t_{5n}^5) + 75(t_{2n}^6 + t_{4n}^4) + t_{6n}^6}{864t_{3n}^3}) \quad (11)$$

The actions of $A$ and $\zeta$ on $E_{\Gamma_n}$ are:

$$A(x, y, t_n) = (-x, \frac{y}{t_n}, \omega_{2n}); \quad (12)$$

$$\zeta(x, y, t_n) = (x, y, \omega_n^{-1}t_n). \quad (13)$$

Therefore, $A$ and $\zeta$ are defined over $\mathbb{Q}(\omega_{2n})$ and $\mathbb{Q}(\omega_n)$ and have order 4 and $n$ respectively. In fact, the $A$ map used in [LLY05] which is defined over $\mathbb{Q}$ is a derivation of the $A$ map here.
Following the notation used before, let
\[ \rho_{k,l,n} : G_\mathbb{Q} \rightarrow \text{Aut}(W_{k,l,n}) \]
be the $l$-adic Scholl representation attached to $S_k(\Gamma_n)$.

3.2. **The space** $S_3(\Gamma_n)$. When $n > 1$, $k = 3$, $j \in (\mathbb{Z}/n\mathbb{Z})^\times$,
\[ \delta_p = \dim_{\mathbb{Q}(\omega_n)} W_j = \dim_{\mathbb{Q}_p(\omega_n)} V_j = 2. \]

For any positive integer $n$, let \( h_j^{[n]} \) be the $l$-adic representation of $G$.

By [ALL05, Prop 2.1], $S_3(\Gamma_n) = \langle h_j^{[n]} \rangle_{j=1}^{n-1}$. It is straightforward to verify that $h_j^{[n]} \in V_j$. According to Theorem 13, we have

**Theorem 14.** For every basis element $h_j^{[n]}$ of $S_3(\Gamma_n)$ and any prime $p > n$, $h_j^{[n]}$ satisfies a three-term Atkin and Swinnerton-Dyer type congruence relation at $p$ given by $\text{Char}(W_j, F_p^{\mathfrak{p}})(T^r)$ as in Lemma 6 where $r = O_n(p)$.

To achieve this result, we only need the operator $\zeta$. When we consider in addition the $A$ map, we will obtain the following result using [Ser77, Prop. 19].

Let $\rho_{3,l,n}^{\text{new}} : G_\mathbb{Q} \rightarrow \text{Aut}(W_{3,l,n}^{\text{new}})$ be the $l$-adic representation attached to the cuspforms genuinely belonging to $\Gamma_n$ as before. When $n = 2, 3, 4$, let $B$ be $A, A$, and $C$; let $K$ be $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$, and $\mathbb{Q}(i)$ respectively. The action of $B$ on $W_{3,l,n}^{\text{new}}$ satisfies $B^2 = -1$. Decompose $W_{3,l,n}^{\text{new}}$ according to $(\pm i)$-eigenspaces of $B$. As a Galois $G_K$-module, $W_{3,l,n}^{\text{new}} = W_{3,l,n,i}^{\text{new}} \oplus W_{3,l,n,-i}^{\text{new}}$. Let $\rho_{3,l,n,\pm i}^{\text{new}} : G_K \rightarrow \text{Aut}(W_{3,l,n,\pm i}^{\text{new}})$. Then we have

**Proposition 15.** When $n = 2, 3, 4$,
\[ \rho_{3,l,n}^{\text{new}} = \text{Ind}^{G_\mathbb{Q}}_{G_K} \rho_{3,l,n,\pm i}^{\text{new}}. \]

**Remark 16.** The above theorem provides a different perspective for the results in [LLY05]. When $n = 2$, each $\rho_{3,l,n,\pm i}^{\text{new}}$ is a one-dimensional representation, hence a character of $G_{\mathbb{Q}(i)}$. Naturally it corresponds to a cuspidal $\eta(4z)^6$ with complex multiplication ([LLY05, section 8]). When $n = 3$, $\rho_{3,l,3}^{\text{new}}$ is induced from a 2-dimensional representation of $G_{\mathbb{Q}(\sqrt{-3})}$. Such a point of view was suggested by J.P. Serre as one of his comments on [LLY05]. A similar result also holds for the $n = 6$ case when another derivation of $A$, called $\tilde{A}$ (14) is used.

4. **$S_3(\Gamma_6)$**

When $k = 3$, $\dim_{\mathbb{Q}(\omega_n)} W_{3,l,n}^{\text{new}} = 2\phi(n)$. Accordingly, only when $n = 2, 3, 4$, or 6 do we have $\dim_{\mathbb{Q}(\omega_n)} W_{3,l,n}^{\text{new}} \leq 4$. Since the first three cases have been handled in [LLY05, ALL05], we will now treat the $n = 6$ case here. The current case under consideration shares a lot of similarities with the case $n = 4$ studied in [ALL05]. To avoid duplication, we will refer the readers to
[ALL05] for some arguments. In this case, we use the following action which is a variation of $\tilde{A}$:

$$\tilde{A}(x, y, t_6) = (-x, y, \frac{i}{t_6}).$$

We use $\tilde{A}$ here since it is defined over a smaller field $\mathbb{Q}(i)$ while $A$ is defined over $\mathbb{Q}(e^{2\pi i/12})$.

Associated with $S_3(\Gamma_6)$ is a 10-dimensional $l$-adic Scholl representation [Sch85]

$$\rho_{3,l,6} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(W_{3,l,6}).$$

In this case, we can pick $N = 6$ (c.f. Section 4.2). As before, the space $W_{3,l,6} \otimes_{\mathbb{Q}_l} \mathbb{Q}_l(\omega_6)$ decomposes naturally into eigenspaces of $\zeta$. Denoted by $W_j$ the $\mathbb{Q}_l(\omega_6)$-eigenspace of $\zeta$ with eigenvalue $\omega_j^n$. Similar to the discussion in Section 5.3 of [ALL05], $W_3$ (resp. $W_2 \oplus W_4$) is isomorphic to the $l$-adic Scholl representation associated with weight 3 cuspforms for $\Gamma_2$ (resp. $\Gamma_3$) tensoring with $\mathbb{Q}_l(\omega_6)$. The remaining piece $W_1 \oplus W_5$, denoted by $W_{\text{new}}$ (or simply $W_{3,l,6}^{\text{new}}$), is a representation space of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (c.f. Cor. 4). By Prop. 7, for every Frobenius element $F_p$ where $p \nmid 6$,

$$\text{Char}(W_{3,l,6}^{\text{new}}, F_p)(T) \in \mathbb{Z}[T].$$

Let $V$ be the $p$-adic Scholl space attached to $S_3(\Gamma_6)$ with the action of Frobenius $F$. We will drop the superscript [6] below for convenience. Decompose $V$ accordingly into $V^{\text{old}} \oplus V^{\text{new}}$ such that

$$h_2, h_3, h_4 \in V^{\text{old}} \text{ and } V^{\text{new}} = V_1 \oplus V_5,$$

where $h_1 \in V_1$, and $h_5 \in V_5$.

By considering $h_j = E_2 \cdot t_n^{n-j}$ and the actions of $\zeta$ and $\tilde{A}$ on $E_1, E_2$, and $t_n$, we obtain

$$h_j|\zeta = \omega_j^n h_j, \quad h_j|\tilde{A} = -i^{n-j} h_{n-j}.$$  

It follows

$$\zeta \tilde{A} \zeta = \tilde{A}. \quad (15)$$

A straightforward computation reveals that

$$F \tilde{A} F^{-1} = \zeta^{3(p-1)/2} \tilde{A}. \quad (16)$$

4.1. **Numerical data.** In this section, we will indicate how the results were discovered numerically.

4.1.1. **$p$-adic side.** Since $h_3$ (resp. $h_2$ and $h_4$) has been discussed in [LLY05] (resp. [ALL05]), we will investigate three-term Atkin and Swinnerton-Dyer congruences for $h_1$ and $h_5$ only. Using the “$p$-adic” Hecke operators mentioned in Section 2.6, A.O.L. Atkin observed that for any odd prime $p > 3$,

- $h_1|\tau_p = c \cdot h_1, \quad h_5|\tau_p = c_2 \cdot h_5$, when $p \equiv 1 (\text{mod } 6)$ with eigenvalues $c$ and $c_2$. 

• $h_1|_{T_p} = c \cdot h_5$, $h_5|_{T_p} = c_2 \cdot h_1$, when $p = 5 \pmod{6}$, where $c = c_2$ if $p = 1 \pmod{4}$ (resp. $c = -c_2$ if $p = -1 \pmod{4}$). Hence $h_1 \pm h_5$ (resp. $h_1 \pm ih_5$) are eigenfunctions with eigenvalues $\pm \sqrt{c \cdot c_2}$.

Here, $s(p) = 1$, if $p = 1 \pmod{12}$ and $s(p) = -1$ otherwise.

In table 1, we list constants $c$ canonically modified as follows:

- when $p = 1 \pmod{12}$, we divide $c$ by $(-3)^{(p-1)/4}$ mod $p = \pm 1$.

Otherwise we divide $c$ by a canonical square root $c_1$ determined as follows:

- when $p = 5 \pmod{12}$, $c_1 = \pm \sqrt{-1} = -(-3)^{(p-1)/4}$ mod $p$;
- when $p = 7 \pmod{12}$, $c_1 = \pm \sqrt{-3} = (-3)^{(p+4)/4}$ mod $p$;
- when $p = 11 \pmod{12}$, $c_1 = \pm \sqrt{3} = (-3)^{(p+1)/4}$ mod $p$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$p \pmod{12}$</th>
<th>modified $c$</th>
<th>$p$</th>
<th>$p \pmod{12}$</th>
<th>modified $c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>7</td>
<td>37</td>
<td>1</td>
<td>-10</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>5</td>
<td>41</td>
<td>5</td>
<td>-50</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>-5</td>
<td>43</td>
<td>7</td>
<td>-10</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>20</td>
<td>47</td>
<td>11</td>
<td>-50</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>-8</td>
<td>53</td>
<td>5</td>
<td>-47</td>
</tr>
<tr>
<td>19</td>
<td>7</td>
<td>-6</td>
<td>59</td>
<td>11</td>
<td>20</td>
</tr>
<tr>
<td>23</td>
<td>11</td>
<td>2</td>
<td>61</td>
<td>1</td>
<td>-64</td>
</tr>
<tr>
<td>29</td>
<td>5</td>
<td>10</td>
<td>67</td>
<td>7</td>
<td>-50</td>
</tr>
<tr>
<td>31</td>
<td>7</td>
<td>31</td>
<td>71</td>
<td>11</td>
<td>0</td>
</tr>
</tbody>
</table>

(Table 1)

4.2. $l$-adic representation side. Like the discussions in [LLY05, Section 5] and [ALL05, Section 3.1], there is an $l$-adic representation $\rho_{3,l,6}^*$ constructed from the elliptic modular surface $E_{16}$. Like [ALL05, Section 3.1], we can show that $\rho_{3,l,6}^*$ is unramified at 5 and $\rho_{3,l,6}$ is isomorphic to $\rho_{3,l,6}^*$. The computation on $\rho_{3,l,6}^*$ is very explicit. By using a Magma program, we obtain the following list:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\text{Char}(W_{\text{new}}, F_p)(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$(T^2 + 7iT - 5)(T^2 - 7iT - 5^2)$ = $T^4 - 14T^2 + 1$</td>
</tr>
<tr>
<td>7</td>
<td>$(T^2 + 5\sqrt{-3T - 7})(T^2 - 5\sqrt{-3T - 7}^2)$ = $T^4 - 23T^2 + 7^4$</td>
</tr>
<tr>
<td>11</td>
<td>$(T^2 - 5\sqrt{-3T - 11})(T^2 + 5\sqrt{-3T - 11}^2)$ = $T^4 - 167T^2 + 11^4$</td>
</tr>
<tr>
<td>13</td>
<td>$(T^2 + 20T + 13^2)^2$</td>
</tr>
<tr>
<td>17</td>
<td>$(T^2 + 8iT - 17)(T^2 - 8iT - 17^2)$ = $T^4 - 514T^2 + 17^4$</td>
</tr>
<tr>
<td>19</td>
<td>$(T^2 + 6\sqrt{-3T - 19})(T^2 - 6\sqrt{-3T - 19}^2)$ = $T^4 - 614T^2 + 19^4$</td>
</tr>
<tr>
<td>23</td>
<td>$(T^2 - 2\sqrt{-3T - 23})(T^2 + 2\sqrt{-3T - 23}^2)$ = $T^4 - 1046T^2 + 23^4$</td>
</tr>
<tr>
<td>37</td>
<td>$(T^2 + 10T + 37^2)^2$</td>
</tr>
<tr>
<td>53</td>
<td>$(T^2 - 47iT - 53)(T^2 + 47iT - 53^2)$ = $T^4 - 3409T^2 + 53^4$</td>
</tr>
</tbody>
</table>

(Table 2)
4.3. **Congruence side.** We have identified a level 108 weight 3 congruence newform by using Magma with the first few terms given as below

\[
\bar{g} := q + \left(\frac{1}{10}u - \frac{17}{10}\right)q^2 + \left(-\frac{1}{5}u - \frac{3}{5}\right)q^4 + 7q^5 + \left(-\frac{1}{2}u + \frac{7}{2}\right)q^7
\]

\[+ 8q^8 + \left(\frac{7}{10}u - \frac{119}{10}\right)q^{10} + \left(\frac{1}{2}u - \frac{7}{2}\right)q^{11} + 20q^{13}\]

\[+ \left(\frac{1}{2}u + \frac{23}{2}\right)q^{14} + \left(\frac{4}{5}u - \frac{68}{5}\right)q^{16} - 8q^{17} + \left(\frac{3}{5}u - \frac{21}{5}\right)q^{19}\]

\[+ \left(-\frac{7}{5}u - \frac{21}{5}\right)q^{20} + \left(-\frac{1}{2}u - \frac{23}{2}\right)q^{22}\]

\[+ \left(-\frac{1}{5}u + \frac{7}{5}\right)q^{23} + \cdots - 10q^{37} + \cdots - 47q^{53} + \cdots\]

\[= \sum_{n \geq 1} a(n)q^n,
\]

where \(u\) is a root of \(x^2 - 14x + 349 = 0\). Different choices of \(u\) give rise to two weight 3 congruence newforms which are the same up to a twisting by \(\chi_{-4}\). We have verified that up to \(p = 541\) the \(p\)th coefficients listed in Table 2 and beyond match the corresponding modified coefficients of the congruence newform (with \(u = 7 - 10\sqrt{-3}\)). In particular, in the comparison, we use \(a(p)/(i\sqrt{3})\) instead of \(a(p)\) when \(p = 3 \mod 4\).

5. **Modularity of \(\rho_{3,l,6}\)**

The reader will find the following section resembles Section 8 of [ALL05]. But the modularity is achieved via a method similar to that used in [LLY05] since Livné’s criterion does not apply to the current case. Moreover, we will focus on the modularity of \(\rho_{3,l,6}'\) in this section.

Let \(\lambda = 1 + i\). Denote by \(\rho_\bar{g}\) the \(l\)-adic Deligne representation attached to the newform \(\bar{g}\) above. Let \(L = \mathbb{Q}(i, \sqrt{3})\). Like [ALL05, Section 8.1], define an ideal class character \(\chi\) by composing the Artin reciprocity map from the idele class group of \(\mathbb{Q}(i)\) to \(\text{Gal}(L/\mathbb{Q}(i))\) with the isomorphism which sends \(\text{Gal}(L/\mathbb{Q}(i))\) to \(\langle i \rangle\). In particular,

- \(\chi(p) = 1\) for any \(p \equiv 3 \mod 4\);
- \(\chi(p) = \pm 1\) for any prime \(v\) above \(p\) when \(p \equiv 1 \mod 12\), depending on the value of \((-3)^{(p-1)/4}\) mod \(v\);
- \(\chi(p) = \pm i\) for any prime \(v\) above \(p\) when \(p \equiv 5 \mod 12\), depending on the value of \((-3)^{(p-1)/4}\) mod \(v\).

Denote by \(\rho_\bar{g}\) the restriction of \(\rho_{\bar{g}}'\) to \(G_{\mathbb{Q}(i)}\). There is a cusp form \(g\) for \(\text{GL}_2(\mathbb{Q}(i))\) corresponding to \(\rho_\bar{g}\). The form \(g\) is the lifting of \(\bar{g}\) over \(\mathbb{Q}(i)\) under the base change by Langlands. Since \(\bar{g}\) and \(\tilde{g}\) twisted by \(\chi_{-4}\) both lift to \(g\), corresponding to \(\rho_g \otimes \chi\) is the cusp form \(g_\chi\), called \(g\) twisted by \(\chi\), for \(\text{GL}_2(\mathbb{Q}(i))\). The local \(p\)-factors of the \(L\)-function \(L(s, g_\chi)\) of \(g_\chi\) are very similar to those for \(f_\chi\) in [ALL05, section 8]. The function \(L(s, g_\chi)\) does not depend on the two possible choices for \(\chi\). Moreover, the local \(p\)-factors
of \( L(s, g_\chi) \) agree with \( \text{Char}(W^{\text{new}}, F_p)(T) \) when \( T \) is replaced by \( p^{-s} \) for all primes \( p \) listed in Table 2.

The action of \( \tilde{A} \) is defined over \( \mathbb{Q}(i) \). When restricted to \( \mathbb{Q}(i) \), \( \rho^{\text{new}}_{3, 1, 6} : G_{\mathbb{Q}} \to \text{Aut}(W^{\text{new}}) \) decomposes into two representations \( \rho^{\text{new}}_{\pm} : G_{\mathbb{Q}(i)} \to \text{Aut}(W^{\text{new}}_{\pm}) \) according to the \((\pm i)\)-eigenspaces of \( \tilde{A} \). Each \( W^{\text{new}}_{\pm} \) is a 2-dimensional \( \mathbb{Q}_{1+i} \)-vector space.

**Theorem 17.** The representation \( W^{\text{new}}_+ \) (or \( W^{\text{new}}_- \)) of \( G_{\mathbb{Q}(i)} \) is isomorphic, up to semisimplification, to the representation space for \( \rho_\tilde{g} \otimes \chi \).

Unlike the case in [ALL05], we cannot apply Livnč’s criterion as the traces are not always even. Like the discussion in [ALL05], we use a special case of the Faltings-Serre’s criterion [Ser]. The ramified places are \( 1 + i \) and 3.

**Theorem 18** (Serre). Let \( \rho_1 \) and \( \rho_2 \) be representations of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i)) \) to \( GL_2(\mathbb{Z}[i]_{1+i}) \). Assume they satisfy the following two conditions:

1. \( \det(\rho_1) = \det(\rho_2) \);
2. the two homomorphisms from \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i)) \) to \( GL_2(\mathbb{F}_2) \), obtained from the reductions of \( \rho_1 \) and \( \rho_2 \) modulo \( 1 + i \), are surjective and equal.

If \( \rho_1 \) and \( \rho_2 \) are not isomorphic, then there exists a pair \((\tilde{G}, t)\), where \( \tilde{G} \) is a quotient of the Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i)) \) isomorphic to either \( S_4 \times \{\pm 1\} \) or \( S_3 \times \{\pm 1\} \), and the map \( t : \tilde{G} \to \mathbb{F}_2 \) has value 0 on the elements of \( \tilde{G} \) of order \( \leq 3 \), and 1 on the other elements.

The idea of Faltings-Serre’s criterion has been recast in [LLY05].

Since \( \lambda = 1 + i \), \( \mathbb{Z}_2(i)/\lambda \mathbb{Z}_2(i) \cong \mathbb{F}_2 \).

**Lemma 19.** As 1-dimensional representations of \( G_{\mathbb{Q}(i)} \), \( \det \rho^{\text{new}}_+ \cong \det \rho_\tilde{g} \otimes \chi \).

**Proof.** We use the idea of the proof of [LLY05, Lemma 5.2]. The only quadratic extensions of \( \mathbb{Q}(i) \) unramified outside of \( 1 + i \) and 3 are \( \mathbb{Q}(i)(\sqrt{d}) \) where \( d = i, (1 + i), (1 - i), 3, 3i, 3(1 + i), 3(1 - i) \). Respectively, \( p = 3 + 2i, 3 + 2i, 3 - 2i, 4 + i, 3 + 2i, 4 - i, 4 + i \) are inert in \( \mathbb{Q}(i)(\sqrt{d}) \). Hence, it suffices to compare the values of the two characters \( \det \rho^{\text{new}}_+ \) and \( \rho_\tilde{g} \otimes \chi \) at primes \( 3 \pm 2i \) and \( 4 \pm i \).

**Lemma 20.** There is only one representation \( \rho \) from \( G_{\mathbb{Q}(i)} \) to \( GL_2(\mathbb{F}_2) \) unramified at \( 1 + i \) and 3 such that the characteristic polynomials of \( \rho(F_p) \) are equal to those given in Table 2 modulo \( \lambda \). Furthermore \( \rho \) is surjective.

**Proof.** Let \( \text{sgn} : SL_2(\mathbb{F}_2) \cong S_3 \to \{\pm 1\} \) be the sign function. The kernel of \( \rho \circ \text{sgn} \) fixes a quadratic field \( \mathbb{Q}(i)(\sqrt{d}) \) of \( \mathbb{Q}(i) \) which is unramified outside of \( 1 + i \) and 3. The possible \( d \) are \( 1, i, 1 + i, 1 - i, 3, 3i, 3 + 3i, 3 - 3i \). By the assumption, the image of \( F_{7+2i} \) under \( \rho \) is \( T^2 + T + 1 \). Hence its image under \( \rho \) is of order 3 and under \( \rho \circ \text{sgn} \) its image is 1. However, when
Let \( d = i, 1 + i, 1 - i, 3, 3 + 3i, 3 - 3i \), either \( F_{7+2i} \) or \( F_{7-2i} \) is inert in \( \mathbb{Q}(i)(\sqrt{d}) \). So the only possible choices are \( d = 1, 3i \).

Now we will show \( \rho \) is surjective. If \( d = 1 \), the kernel of \( \rho \circ \text{sgn} \) fixes a cyclic cubic Galois extension \( M \) of \( \mathbb{Q}(i) \) which is unramified outside of \( 1 + i \) and \( 3 \). Hence \( \text{Gal}(M/\mathbb{Q}) = \mathbb{Z}_6 \) or \( S_3 \). Since the order of \( \rho(F_7) \) is at least 6, hence \( \text{Gal}(M/\mathbb{Q}) = \mathbb{Z}_6 \) where \( M = \mathbb{Q}(i)(\omega_9 + \omega_9^{-1}) \) is the splitting field of \( f(x) = x^3 - 3x + 1 \) over \( \mathbb{Q}(i) \). The polynomial \( f(x) \) is irreducible modulo \( 3 + 2i \). Meanwhile, the characteristic polynomial of \( F_{3+2i} \) on \( W^{new} \) modulo \( \lambda = T^2 + 1 \). Its order should be 1 or 2. This leads to a contradiction.

Finally assume that \( \ker(\rho \circ \text{sgn}) = \text{Gal}(\mathbb{Q}/K) \) for some \( S_3 \) extension of \( \mathbb{Q}(i) \) which contains \( \mathbb{Q}(i)(\sqrt{3i}) \). I.e. \( K \) is the splitting field of an irreducible cubic polynomial \( f(x) = x^3 + px + q \), \( p, q, \in \mathbb{Z}[i] \) such that a) the discriminant \( -4p^3 - 27q^2 = \pm(3i)(1 + i)\alpha^3 \beta^3, \alpha, \beta \in \mathbb{Z}_{\geq 0}; \) b) \( f(x) \) is reducible modulo \( 3 + 2i, 6 + i \) (their characteristic polynomials in the residue field are \( T^2 + 1 \) and hence of order at most 2) and is irreducible when modulo \( 7 + 2i \) (its characteristic polynomial in the residue field is \( T^2 + T + 1 \) and hence of order 3). Such a field can be determined uniquely and is the splitting field of \( f(x) = x^3 - 3x - 2i \). So \( \rho \) is unique up to isomorphism.

\[ \square \]

It is easy to see \( \rho^{new}_+ \) and \( \rho^{new}_- \) satisfy the conditions for \( \rho \). Consequently, when modulo \( 1 + i \), the representations \( \rho^{new}_+ \) and \( \rho^{new}_- \) are surjective on \( GL_2(\mathbb{F}_2) \) and isomorphic up to semisimplification.

We now find all \( S_4 \) extensions \( E \) of \( \mathbb{Q}(i) \) which contain \( K \) and are unramified outside away from \( (1 + i) \) and \( 3 \). There is a unique order 48 group which has a normal subgroup isomorphic to \( S_4 \), which is \( S_4 \times \mathbb{Z}_2 \) (The group \( S_4 \times \mathbb{Z}_2 \) is isomorphic to \( S_4 \times \mathbb{Z}_2 \) since all automorphisms of \( S_4 \) are inner). Consequently, each \( S_4 \) extension of \( \mathbb{Q}(i) \) corresponds to a unique \( S_4 \) extension \( L_4 \) of \( \mathbb{Q} \) which is unramified away from \( 2 \) and \( 3 \). Furthermore since \( S_4 \times \mathbb{Z}_2 \) has a unique normal subgroup isomorphic to \( S_3 \), \( E \) has a unique subfield \( L_3 \) which is Galois over \( \mathbb{Q} \) with \( \text{Gal}(L_3/\mathbb{Q}) \cong S_3 \). Over \( \mathbb{Q}(i) \), \( f(x) \) can be obtained from \( h(x) = x^3 + 3x - 2 \) (as \((-x/i)^3 + (-3/i)x - 2 = -i(x^3 - 3x - 2i))\). The polynomial \( h(x) \) appeared in the proof of [LLY05, Lemma 6.2]. Thus all such \( S_4 \) extensions \( L_4 \) of \( \mathbb{Q} \) unramified away from \( 2 \) and \( 3 \) and containing the splitting field of \( h(x) \) are listed in [LLY05, pp. 138]. Accordingly, the following are the quartic polynomials which give rise to \( S_4 \) extensions of \( \mathbb{Q}(i) \) that contain \( K \), unramified outside of \( (1 + i) \) and \( 3 \).

<table>
<thead>
<tr>
<th>defining equation</th>
<th>discriminant</th>
<th>( v \text{ with order } 4 \text{ Frobenius} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^4 - 4ix - 3 = 0 )</td>
<td>((3i)(1 + i)^{18}(3)^2)</td>
<td>( 3 + 2i )</td>
</tr>
<tr>
<td>( x^4 - 8ix + 6 = 0 )</td>
<td>(-(3i)(1 + i)^{22}(3)^2)</td>
<td>( 3 + 2i )</td>
</tr>
<tr>
<td>( x^4 + 12x^2 - 16ix + 12 = 0 )</td>
<td>((3i)(1 + i)^{34}(3)^2)</td>
<td>( 6 + i )</td>
</tr>
</tbody>
</table>
To conclude the claim of Theorem 17, it suffices to compare the local Euler-\(p\) factors of \(L\)-functions attached to the two representations at \(p = 13, 37, 53\).

5.1. The three-term Atkin and Swinnerton-Dyer congruences. We have remarked that in this case \(F \tilde{A}F^{-1} = \zeta^{3(p-1)/2} \tilde{A}\) and \(F \zeta = \zeta^p \tilde{F}\). Let

\[ B_{-1} = \tilde{A}, \quad B_{-3} = (\zeta - \zeta^{-1}), \quad B_3 = \tilde{A}(\zeta - \zeta^{-1}). \]

They satisfy that \(B_{-1}, B_{-3}\), and \(B_3\) are defined over \(\mathbb{Q}_p\) when \(p = 5, 7, 11\) modulo 12 respectively. Moreover on \(V_{\text{new}}\),

\[ B_{-1}^2 = -1, \quad B_{-3}^2 = B_3^2 = -3. \]

Now apply the analysis in Section 7 of [ALL05] with the roles of \(B_{\pm 2}\) replaced by \(B_{\pm 3}\) and \(A\) replaced by \(\tilde{A}\), we obtain the following theorem.

**Theorem 21** (Atkin and Swinnerton-Dyer congruence for \(S_3(\Gamma_6)\)). The Atkin and Swinnerton-Dyer congruence holds on the space \(S_3(\Gamma_6) = \langle h_1, h_2, h_3, h_4, h_5 \rangle\). More precisely, \(h_3\) lies in \(S_3(\Gamma_2)\) and it satisfies the Atkin and Swinnerton-Dyer congruence relations with the congruence form \(g_2(z) = \eta(4z)^6; \ h_2, h_4 \in S_3(\Gamma_3)\) and \(h_2 \pm ih_4\) satisfy Atkin and Swinnerton-Dyer congruence relations with level 27 congruence newforms (as in [LLY05]). For each odd prime \(p \nmid 6l\), the subspace \(\langle h_1, h_5 \rangle\) has a basis depending on the residue of \(p\) modulo 12 satisfying a 3-term Atkin and Swinnerton-Dyer congruence at \(p\) as follows.

1. If \(p \equiv 1 \mod 12\), then both \(h_1\) and \(h_3\) satisfy the 3-term Atkin and Swinnerton-Dyer congruence at \(p\) given by \(T^2 - A(p)T + p^2\);
2. If \(p \equiv 5 \mod 12\), then \(h_1\) (resp. \(h_3\)) satisfies the 3-term Atkin and Swinnerton-Dyer congruence at \(p\) given by \(T^2 + A(p)iT - p^2\) (resp. \(T^2 - A(p)iT - p^2\));
3. If \(p \equiv 7 \mod 12\) (resp. \(p \equiv 11 \mod 12\)), then \(h_1 \pm h_3\) (resp. \(h_1 \pm ih_3\)) satisfy the 3-term Atkin and Swinnerton-Dyer congruence at \(p\) given by \(T^2 \pm \sqrt{-3}A(p)T - p^2\), respectively.

All coefficients \(A(p) \in \mathbb{Z}\) can be derived from the coefficients of \(\tilde{g}\) via the modification described in subsection 4.3.

6. Acknowledgements

The author is indebted to A.O.L. Atkin for his many original observations and insightful comments. As a matter of fact, his discoveries led to the discussions in section 4. The author would also like to thank Wenching W. Li and Siu-Hung Ng for their continuous valuable and enlightening discussions and Chris Kurth for his comments on an earlier version of this paper. Special thanks to William A. Stein for computation assistance.

\[\text{\textsuperscript{1}}B_{\mp}\text{ in the 2005 version of [ALL05]}\]
References


Department of Mathematics, Iowa State University, Ames, IA 50011, USA