Chapter 10

• #2. Let $\phi(x) = |x|$. Since $|xy| = |x| \cdot |y|$, $\phi(x)$ is a homomorphism.

• #5. It is easy to see when $r > 1$, $\phi(x) = x^r$ is not one-to-one. For example, let $\mu_r = \frac{2\pi i}{r}$ when $r > 1$. Then $\phi(x) = \phi(\mu_r x)$. Hence for any positive integer $r$, $x \rightarrow x^r$ is an isomorphism if and only if $r = 1$.

• #7 see solution on Page A15

• #8 for any $\sigma_1, \sigma_2 \in G$, if $\sigma_1$ is an even permutation, then the parity of $\sigma_2$ and $\sigma_1 \sigma_2$ are the same, hence $sgn(\sigma_1 \sigma_2) = 1 \cdot sgn(\sigma_2) = sgn(\sigma_1) sgn(\sigma_2)$.

Similarly, when $\sigma_1$ is odd, then the parity of $\sigma_2$ and $\sigma_1 \sigma_2$ are opposite. Hence $sgn(\sigma_1 \sigma_2) = -1 \cdot sgn(\sigma_2) = sgn(\sigma_1) sgn(\sigma_2)$.

In both cases, we have $sgn(\sigma_1 \sigma_2) = sgn(\sigma_1) sgn(\sigma_2)$. Hence $sgn$ is a homomorphism. The neutral element of the multiplicative group $\{\pm 1\}$ is 1. Hence the kernel of $sgn$ consists of all even permutations.

• #9, Let $\phi(g, h) = g$. Then for any two elements $(g_1, h_1), (g_2, h_2) \in G \oplus H$, we have

$$\phi((g_1, h_1) \cdot (g_2, h_2)) = \phi((g_1 g_2, h_1 h_2)) = g_1 g_2 = \phi((g_1, h_1)) \cdot \phi((g_2, h_2)).$$

Hence the map $\phi$ is a homomorphism. The kernel consists of elements $\{(e, h) | h \in H\}$.

• #16, Since $|\mathbb{Z}_8 \oplus \mathbb{Z}_2| = |\mathbb{Z}_4 \oplus \mathbb{Z}_4| = 16$. If there is an onto homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, then the map is also one-to-one. Hence such a homomorphism is indeed an isomorphism. However, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ are not isomorphic as the first group has order 8 elements and the second group does not.

• #19. For any homomorphism, the map is $n$-to-one, where $n$ divides the order of the first group, which is 17 in this case. Hence, if the homomorphism is not one-to-one, then it is 17-to-one. So the map $\phi$ is the trivial map in the sense that for any $n \in \mathbb{Z}_{17}, \phi(n) = e$.

• #24. The group $\mathbb{Z}_4$ is a cyclic group, hence there are 4 homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which can be specified by the image of the generated 1. $\phi_1(1) = (0, 0), \phi_2(1) = (1, 0), \phi_3(1) = (0, 1), \phi_4(1) = (1, 1)$.

Chapter 11

• #1. 4

• #2. 8

• #5. There are two non-isomorphic abelian groups of order 45, $\mathbb{Z}_5 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Then $(1, 3)$ in the first group and $(1, 1, 1)$ in the second group have order 15. In the second group, there is no order 9 element.

• #15. see solution on Page A16.