SELECTED HOMEWORK SOLUTIONS FOR CHAPTER 1

Section 2

3. (b) First we show that \(f^{-1}(\bigcup_{B \in \mathcal{B}} B) \subseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B)\). If \(x \in f^{-1}(\bigcup_{B \in \mathcal{B}} B)\), then \(f(x) \in \bigcup_{B \in \mathcal{B}} B\), so \(f(x) \in B_0\) for some \(B_0 \in \mathcal{B}\), so \(x \in f^{-1}(B_0) \subseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B)\).

Next we show that \(f^{-1}(\bigcup_{B \in \mathcal{B}} B) \supseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B)\). If \(x \in \bigcup_{B \in \mathcal{B}} f^{-1}(B)\), then there is a \(B_0 \in \mathcal{B}\) such that \(f(x) \in B_0 \subseteq \bigcup_{B \in \mathcal{B}} B\), and hence \(x \in f^{-1}(B_0) \subseteq \bigcup_{B \in \mathcal{B}} f^{-1}(B)\).

(c) If \(x \in f^{-1}(\bigcap_{B \in \mathcal{B}} B)\), then \(f(x) \in \bigcap_{B \in \mathcal{B}} B\), so \(f(x) \in B\) for every \(B \in \mathcal{B}\) and hence \(x \in f^{-1}(B)\) for all \(B\), so \(x \in \bigcap_{B \in \mathcal{B}} f^{-1}(B)\).

Conversely, if \(x \in \bigcap_{B \in \mathcal{B}} f^{-1}(B)\), then, for every \(B \in \mathcal{B}\), we have \(f(x) \in B\), so \(f(x) \in \bigcap_{B \in \mathcal{B}} B\) and \(x \in f^{-1}(\bigcap_{B \in \mathcal{B}} B)\).

(f) If \(y \in f(\bigcup_{A \in \mathcal{A}} A)\), then there is \(x \in \bigcup_{A \in \mathcal{A}} A\) such that \(f(x) = y\), so \(f(x) \in f(A_0)\) for some \(A_0 \in \mathcal{A}\), and therefore \(f(x) \in \bigcup_{A \in \mathcal{A}} f(A)\).

If \(y \in \bigcup_{A \in \mathcal{A}} f(A)\), then there is some \(A_0 \in \mathcal{A}\) such that \(y \in f(A_0)\). Then \(y \in f(\bigcup_{A \in \mathcal{A}} A)\).

(g) If \(y \in f(\bigcap_{A \in \mathcal{A}} A)\), then \(y = f(x)\) for some \(x \in \bigcap_{A \in \mathcal{A}} A\). Hence \(x \in A\) for all \(A \in \mathcal{A}\) so \(y \in f(A)\) for all \(A \in \mathcal{A}\) and therefore \(y \in \bigcap_{A \in \mathcal{A}} f(A)\).

If \(y \in \bigcap_{A \in \mathcal{A}} f(A)\) and \(f\) is injective, then, for each \(A \in \mathcal{A}\), there is an \(x_A \in A\) such that \(f(x_A) = y\). Because \(f\) is injective, there is only one \(x \in X\) such that \(f(x) = y\) and this \(x\) is in every \(A\), so \(x \in \bigcap_{A \in \mathcal{A}} A\) and \(y \in f(\bigcap_{A \in \mathcal{A}} A)\).

Section 3

3. The flaw in this argument is that \(a\), \(b\) and \(c\) are not quantified. The variable \(a\) should be described as “for every \(a \in A\)” but \(b\) is “there exists \(b \in A\).” For some symmetric and transitive relations, it is true that for any \(a \in A\), there is \(b \in A\) such that \(aCb\) but this situation does not always occur.

Section 7

5. (a) This set is countable. For each \(f \in A\), define \(g(f) = (f(0), f(1))\). Then \(g\) is a bijection onto \(\mathbb{Z}_+ \times \mathbb{Z}_+\), which is countable.

(e) This set is uncountable. We’ll show that given any sequence \(f_1, f_2, \ldots\) of functions in \(E\), there is a function \(f \in E\) which is not in the sequence. Specifically, we define \(f\) by

\[
f(i) = \begin{cases} 
0 & \text{if } f_i(i) = 1 \\
1 & \text{if } f_i(i) = 0.
\end{cases}
\]

(f) This set is countable. For each \(n \in \mathbb{Z}_+ \cup \{0\}\), define the set

\[F_n = \{f \in F | f(m) = 0 \text{ if } m \geq n\}.
\]

Then each \(F_n\) is finite; in fact it has exactly \(2^n\) elements. In addition \(F = \cup_{n=0}^{\infty} F_n\), so \(F\) is a countable union of finite sets and therefore it’s countable.
Section 9

4. Theorem 7.5 used the Axiom of Choice to choose the $f_n$s. In fact, the proof uses the phrase “we can choose, for each $n$, a surjective function $f_n$”. Rather than use the Axiom of Choice directly, I will use the version from class, which is Lemma 9.2. We define $B_n$ to be the set of all surjective functions from $\mathbb{Z}^+$ onto $A_n$ and then $\mathcal{B}$ is the collection of all the $B_n$s. Then there is a choice function $c: \mathcal{B} \rightarrow \bigcup_{n \in \mathbb{Z}^+} B_n$ such that $c(B_n) \in B_n$ for all $n \in \mathbb{Z}^+$. We now set $f_n = c(B_n)$. A similar argument gives a surjective function $g: \mathbb{Z}^+ \rightarrow J$. With this construction for $f_n$ and $g$, the proof continues as in the book.

Section 10

3. They do not have the same order type. As I mentioned in class, the set

$$\{x \in \{1, 2\} \times \mathbb{Z}^+ | x < 2 \times 1\}$$

is infinite, but, for any $b \in \mathbb{Z}^+ \times \{1, 2\}$, the set

$$\{x \in \{1, 2\} \times \mathbb{Z}^+ | x < b\}$$

is finite. Therefore there cannot be a bijection $f: \{1, 2\} \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \{1, 2\}$ which preserves order.

7. We prove the theorem by contradiction. If $J_0$ is an inductive subset of $J$ which is not all of $J$, then $J - J_0$ has a least element $\alpha$. But then $\beta < \alpha$ implies $\beta \in J_0$, so $S_\alpha \subset J_0$. Because $J_0$ is inductive, it follows that $\alpha \in J_0$, which contradicts our assertion that $\alpha \in J - J_0$. 