Math 534
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October 11, 2001

SELECTED HOMEWORK SOLUTIONS FOR CHAPTER 1

Section 2

3. (b) First we show that $f^{-1}(\cup_{B\in\mathcal{B}}B) \subset \cup_{B\in\mathcal{B}}f^{-1}(B)$. If $x \in f^{-1}(\cup_{B\in\mathcal{B}}B)$, then $f(x) \in \cup_{B\in\mathcal{B}}$, so $f(x) \in B_0$ for some $B_0 \in \mathcal{B}$, so $x \in f^{-1}(B_0) \subset \cup_{B\in\mathcal{B}}f^{-1}(B)$.
   Next we show that $f^{-1}(\cup_{B\in\mathcal{B}}B) \supset \cup_{B\in\mathcal{B}}f^{-1}(B)$. If $x \in \cup_{B\in\mathcal{B}}f^{-1}(B)$, then there is a $B_0 \in \mathcal{B}$ such that $f(x) \in B_0 \subset \cup_{B\in\mathcal{B}}B$, and hence $x \in f^{-1}(\cup_{B\in\mathcal{B}}, B)$.
   (c) If $x \in f^{-1}(\cap_{B\in\mathcal{B}})$, then $f(x) \in \cap_{B\in\mathcal{B}}B$, so $f(x) \in B$ for every $B \in \mathcal{B}$ and hence $x \in f^{-1}(B)$ for all B, so $x \in \cap_{B\in\mathcal{B}}f^{-1}(B)$.
   Conversely, if $x \in \cap_{B\in\mathcal{B}}f^{-1}(B)$, then, for every $B \in \mathcal{B}$, we have $f(x) \in B$, so $f(x) \in \cap_{B\in\mathcal{B}}B$ and $x \in f^{-1}(\cap_{B\in\mathcal{B}}B)$.
   (f) If $y \in f(\cup_{A\in\mathcal{A}}A)$, then there is $x \in \cup_{A\in\mathcal{A}}A$ such that $f(x) = y$, so $f(x) \in f(A_0)$ for some $A_0 \in \mathcal{A}$, and therefore $f(x) \in \cup_{A\in\mathcal{A}}f(A)$.
   If $y \in \cup_{A\in\mathcal{A}}f(A)$, then there is some $A_0 \in \mathcal{A}$ such that $y \in f(A_0)$. Then $y \in f(\cup_{A\in\mathcal{A}}A)$.
   (g) If $y \in f(\cap_{A\in\mathcal{A}}A)$, then $y = f(x)$ for some $x \in \cap_{A\in\mathcal{A}}A$. Hence $x \in A$ for all $A \in \mathcal{A}$ so $y \in f(A)$ for all $A \in \mathcal{A}$ and therefore $y \in \cap_{A\in\mathcal{A}}f(A)$.
   If $y \in \cap_{A\in\mathcal{A}}f(A)$ and f is injective, then, for each $A \in \mathcal{A}$, there is an $x_A \in A$ such that $f(x_A) = y$. Because f is injective, there is only one $x \in X$ such that $f(x) = y$ and this x is in every A, so $x \in \cap_{A\in\mathcal{A}}A$ and $y \in f(\cap_{A\in\mathcal{A}}A)$.

Section 3

3. The flaw in this argument is that a, b and c are not quantified. The variable a should be described as "for every $a \in A$" but b is "there exists $b \in A$". For some symmetric and transitive relations, it is true that for any $a \in A$, there is $b \in A$ such that aCb but this situation does not always occur.

Section 7

5. (a) This set is countable. For each $f \in A$, define $g(f) = (f(0), f(1))$. Then g is a bijection onto $\mathbb{Z}_+ \times \mathbb{Z}_+$, which is countable.
   (e) This set is uncountable. We'll show that given any sequence $f_1, f_2, \dots$ of functions in E, there is a function $f \in E$ which is not in the sequence. Specifically, we define f by
$$f(i) = \begin{cases} 0 & \text{if } f_i(i) = 1 \\ 1 & \text{if } f_i(i) = 0. \end{cases}$$
   (f) This set is countable. For each $n \in \mathbb{Z}_+ \cup \{0\}$, define the set
$$F_n = \{f \in F | f(m) = 0 \text{ if } m \geq n\}.$$
   Then each F_n is finite; in fact it has exactly 2^n elements. In addition $F = \cup_{n=0}^{\infty}F_n$, so F is a countable union of finite sets and therefore it's countable.

Section 9

4. Theorem 7.5 used the Axiom of Choice to choose the f_ns. In fact, the proof uses the phrase "we can choose, for each n, a surjective function f_n". Rather than use the Axiom of Choice directly, I will use the version from class, which is Lemma 9.2. We define B_n to be the set of all surjective functions from $\mathbb{Z}_+$ onto A_n and then $\mathcal{B}$ is the collection of all the B_ns. Then there is a choice function $c\colon \mathcal{B} \to \cup_{n \in \mathbb{Z}_+} B_n$ such that $c(B_n) \in B_n$ for all $n \in \mathbb{Z}_+$. We now set $f_n = c(B_n)$. A similar argument gives a surjective function $g\colon \mathbb{Z}_+ \to J$. With this construction for f_n and g, the proof continues as in the book.

Section 10

3. They do not have the same order type. As I mentioned in class, the set
$$\{x \in \{1, 2\} \times \mathbb{Z}_+ | x < 2 \times 1\}$$
is infinite, but, for any $b \in \mathbb{Z}_+ \times \{1, 2\}$, the set
$$\{x \in \{1, 2\} \times \mathbb{Z}_+ | x < b\}$$
is finite. Therefore there cannot be a bijection $f\colon \{1, 2\} \times \mathbb{Z}_+ \to \mathbb{Z}_+ \times \{1, 2\}$ which preserves order.

7. We prove the theorem by contradiction. If J_0 is an inductive subset of J which is not all of J, then $J - J_0$ has a least element α. But then $\beta < \alpha$ implies $\beta \in J_0$, so $S_\alpha \subset J_0$. Because J_0 is inductive, it follows that $\alpha \in J_0$, which contradicts our assertion that $\alpha \in J - J_0$.