SELECTION HOMEWORK #3 SOLUTIONS

Section 13
1. For each \( x \in A \), write \( U_x \) for the open set such that \( x \in U_x \subset A \). Then
\[
\bigcup_{x \in A} U_x \subset A
\]
because \( U_x \subset A \). On the other hand
\[
A \subset \bigcup_{x \in A} U_x
\]
because for each \( x \in A \), we have \( x \in U_x \). It follows that \( \bigcup_{x \in A} U_x = A \), so \( A \) is a union of open sets, and hence \( A \) is open.

4. (a) It’s easy to check that \( \mathcal{T} \) is a topology. To see that \( \bigcup_{\alpha \in J} \mathcal{T}_\alpha \) need not be a topology, let \( X = \{a, b, c\} \) with
\[
\mathcal{T}_1 = \{\emptyset, \{a\}, X\}, \quad \mathcal{T}_2 = \{\emptyset, \{b\}, X\}.
\]
Then
\[
T_1 \cup T_2 = \{\emptyset, \{a\}, \{b\}, X\}
\]
is not a topology because \( \{a\} \cup \{b\} = \{a, b\} \) isn’t in \( T_1 \cup T_2 \).
(b) Let \( \mathcal{S} = \bigcup_{\alpha \in J} \mathcal{T}_\alpha \). Then \( \mathcal{S} \) is a subbasis for a topology \( \mathcal{T} \), and clearly \( \mathcal{S} \subset \mathcal{T} \). On the other hand, if \( \mathcal{T}' \) is any topology with \( \mathcal{S} \subset \mathcal{T}' \), then the construction of \( \mathcal{T} \) implies that \( \mathcal{T} \subset \mathcal{T}' \). Hence \( \mathcal{T} \) is the smallest topology containing all \( \mathcal{T}_\alpha \).

Since \( \bigcap_{\alpha \in J} \mathcal{T}_\alpha \) is a topology (by part (a)), it’s the smallest topology contained in all \( \mathcal{T}_\alpha \).

Section 16
7. Let \( X = \mathbb{R} - \{0\} \) and let \( Y \) be the set of positive real numbers, using the usual ordering.
Then \( Y \) is clearly convex, but \( Y \) is not an interval in \( X \) because there is no number \( b \) satisfying \( b \geq x \) for all \( x \in Y \). In addition \( Y \) is not a ray because it would have to have the form \( Y = (a, +\infty) \) with \( a \in X \), in which case \( a < y \) for all \( y \in Y \). But \( a \) can’t be positive (because \( a \in Y \)) and \( a \) can’t be negative because then \( a/2 \in (a, y) \) for all \( y \in Y \). Therefore, a convex set need not be an interval or a ray.