TAKE HOME FINAL EXAM

DIRECTIONS: Turn in solutions to 9 problems, at least 5 of which are chosen from problems 1-10 and at least 2 of which are from 11-17. No more than 9 solutions will be graded. The solutions are due at 11:45 am, Monday, December 17.

CHANGES:
December 3:
Problem 1: change sup to inf. Also some corrections of punctuation.
Problem 2: change $K : \mathcal{P}(X) \to (X)$ to $K : \mathcal{P}(X) \to \mathcal{P}(X)$, change $K(\{x\}) = x$ to $K(\{x\}) = \{x\}$, change $\alpha in J$ to $\alpha \in J$, change $K(A) \subset A$ to $A \subset K(A)$.

December 5:
Problem 5: change $|x - y|$ to $|x - y|^\alpha$ in the definition of $S(\alpha, f)$
Problem 7: add the word “and” before $fg \in A$.
Problem 13: add “or #2”
add Problem 17

December 6:
Problem 6: The $f_n$s should be continuous.

1. A function $\omega : \mathbb{R} \to \mathbb{R}$ is called subadditive if $\omega(x + y) \leq \omega(x) + \omega(y)$ for all $x$ and $y$ in $\mathbb{R}$. If $X$ is a metric space and $f : X \to \mathbb{R}$ is uniformly continuous, a modulus of continuity for $f$ is a continuous, nondecreasing, subadditive function $\omega$ such that $\omega(0) = 0$ and

$$|f(x) - f(y)| \leq \omega(d(x, y))$$

for all $x, y$ in $X$.

(a) If $A$ is a compact subset of $X$ and $f : A \to \mathbb{R}$ is uniformly continuous with modulus of continuity $\omega$, show that $g : X \to \mathbb{R}$ defined by

$$g(x) = \inf \{f(y) + \omega(d(x, y)) \mid y \in A\}$$

is a uniformly continuous extension of $f$ with modulus of continuity $\omega$. (b) If $X$, $A$, $f$, $\omega$ are as in part (a) and if there are real numbers $a$ and $b$ such that $f(X) \subset [a, b]$, use part (a) to find a uniformly continuous extension $G : X \to [a, b]$ of $f$ with modulus of continuity $\omega$.

2. Let $X$ be a set and suppose there is a function $K : \mathcal{P}(X) \to \mathcal{P}(X)$ such that

$K(A \cup B) = K(A) \cup K(B)$ for all subsets $A$ and $B$ of $X$,

$K(\emptyset) = \emptyset$,

$K(\{x\}) = \{x\}$ for all $x \in X$,

$K(K(A)) = K(A)$ for all $A \subset X$.

(a) Show that $A \subset B$ implies that $K(A) \subset K(B)$ and that $K(A) - K(B) \subset K(A - B)$ for any subsets $A$ and $B$ of $X$. 

3. Let \((X, \mathcal{T})\) be a topological space. Prove that this space is Hausdorff if and only if \(D = \{x \times x \mid x \in X\}\) is closed in \(X \times X\).

4. Show that \(S_a\) and \(\overline{S_a}\) are metrizable for any \(a \in S_{\Omega}\).

5. For \(\alpha \in (0, 1)\) and \(f \in C([0, 1], \mathbb{R})\), define
\[
S(\alpha, f) = \{H \mid |f(x) - f(y)| \leq H |x - y|^\alpha, |f(x)| \leq H \text{ for all } x, y \in [0, 1]\},
\]
\[
|f|_\alpha = \inf S(\alpha, f), \quad d_\alpha(f, g) = |f - g|_\alpha, \quad C^\alpha = \{f \mid S(\alpha, f) \neq \emptyset\}.
\]
(a) Show that \(C^\alpha \subset C^\beta\) if \(0 < \beta < \alpha < 1\).
(b) If \(0 < \alpha < \beta < 1\), prove that \(\{f \mid |f|_\beta \leq 1\}\) is a compact subset of \(C^\alpha\).
(c) Show that \((C^\alpha, d_\alpha)\) is complete for \(0 < \alpha < 1\).
(d) Show that \(C^\alpha\) is dense in \(C([0, 1])\).

6. (a) Prove Dini’s theorem: Let \((f_n)\) is a sequence of continuous real-valued functions on the interval \([0, 1]\) and suppose that \(f_n(x) \to 0\) for each \(x \in [0, 1]\) and \(f_n(x) \geq f_{n+1}(x)\) for each \(x \in [0, 1]\). Then \(f_n \to 0\) uniformly. (Hint: Let \((x_n)\) be a sequence of points such that \(f_n\) takes on its maximum value at \(x_n\) and let \(\varepsilon > 0\). First, show that a subsequence \((x_{nk})\) converges to some \(x_0 \in [0, 1]\) and then that, if \(f_N(x_0) < \varepsilon\), there is a \(\delta > 0\) such that \(f_n(x) < \varepsilon\) for all \(n \geq N\) and all \(x \in (x_0 - \delta, x_0 + \delta)\). What does this say about \(\max f_{nk}\) for \(k\) sufficiently large?)
(b) Use Dini’s theorem to show that the sequence of polynomials \((p_n)\) defined inductively by
\[
p_1(t) = t/2, \quad p_{n+1}(t) = p_n(t) + (t - p_n(t)^2)/2
\]
converges uniformly to the square root function \(\sqrt{t}\) on \([0, 1]\).

7. This question is about an approximation theorem called the Stone-Weierstrass Theorem. Let \(X\) be a compact Hausdorff space and give \(C(X, \mathbb{R})\) the uniform metric. A subset \(\mathcal{A}\) of \(C(X, \mathbb{R})\) is called a subalgebra if \(f\) and \(g\) in \(\mathcal{A}\) imply \(af + bg \in \mathcal{A}\) (for all real number \(a\) and \(b\)) and \(fg \in \mathcal{A}\). We say that \(\mathcal{A}\) separates points if for any two distinct points \(x\) and \(y\) in \(X\), there is a function \(f \in \mathcal{A}\) such that \(f(x) \neq f(y)\). The Stone-Weierstrass Theorem states that if \(\mathcal{A}\) is a subalgebra which is closed, which separates points and contains all constant functions, then \(\mathcal{A} = C(X, \mathbb{R})\). Prove this theorem. Hints:
(b) If \(\mathcal{A}\) is a subalgebra which is closed and if \(f \in \mathcal{A}\), then \(|f| \in \mathcal{A}\). (Problem 6b may be helpful. You may quote that problem without solving it.)
(c) If \( \mathcal{A} \) is a subalgebra which is closed and if \( f \) and \( g \) are in \( \mathcal{A} \), then so is \( \max\{f, g\} \).

(d) If \( \mathcal{A} \) is a subalgebra which separates points and contains the constants, then for all distinct \( x \) and \( y \) in \( X \) and all \( a \) and \( b \) in \( \mathbb{R} \), there is \( f \in \mathcal{A} \) such that \( f(x) = a \), \( f(y) = b \).

(e) Fix \( \varepsilon > 0 \) and \( f \in C(X, \mathbb{R}) \). Let \( \mathcal{A} \) be a subalgebra which is closed and which separates points. For each distinct \( x \) and \( y \) in \( X \), let \( u_{x,y} \) be an element of \( \mathcal{A} \) such that \( u_{x,y}(x) = f(x) \) and \( u_{x,y}(y) = f(y) \). Show that the set

\[
G_y = \{ z \in X \mid u_{x,y}(z) > f(z) - \varepsilon \}
\]

is open and use the compactness of \( X \) along with part (c) to find, for each \( y \in X \), a function \( u_y \in \mathcal{A} \) such that \( u_y(z) > f(z) - \varepsilon \) for all \( z \in X \) and \( u_y(y) = f(y) \).

(f) Imitate part (e) to obtain \( g \in \mathcal{A} \) such that \( \sup\{|f(z) - g(z)| \mid z \in X\} < \varepsilon \).

(g) Conclude that \( \mathcal{A} = C(X, \mathbb{R}) \).

8. (a) Show that the Stone-Weierstrass Theorem implies the classical Weierstrass Approximation Theorem: If \( K \) is a cube in \( \mathbb{R}^n \), then for any \( f \in C(K, \mathbb{R}) \) and any positive \( \varepsilon \), there is a polynomial \( p \) such that \( |p - f| < \varepsilon \) on \([0,1] \).

(b) Give an example to show that the Stone-Weierstrass Theorem need not be true if \( X \) is not compact.

(c) What happens to the Stone-Weierstrass Theorem if \( X \) is not Hausdorff?

9. A space is \( \sigma \)-compact if it is the union of countably many compact subsets.

(a) Prove that every \( \sigma \)-compact space is Lindelöf, but not conversely.

(b) Prove that \( S_\Omega \times I^I \) is countably compact but not \( \sigma \)-compact.

(c) Find a \( \sigma \)-compact space which is not countably compact.

10. Show that the space \( I^I \) is compact but not sequentially compact. (Theorem 37.3 may be quoted here.)

PROBLEMS FROM THE TEXTBOOK

11. Section 7, #9

12. Section 17, #21 (Hint: First show that \( \text{Int}(\text{Int}(A)) = \text{Int}(A) \).)

13. Section 32, #2 or #7

14. Section 32, #9

15. Section 43, #4

16. Section 43, #5

17. Section 33, #4 or #8