Math 511
Professor Lieberman
June 2, 2003

MIDTERM EXAM

Directions: Answer seven questions under the additional constraints.

(1) Do at least one problem from each chapter in the text out of the list below. (Note that each line represents one chapter.)
(2) Do five problems from the text.
(3) Do two of the three problems not from the text.
(4) Do not consult with other people except me about the problems on this test. It is, however, permissible to ask questions about the test in class.

From the text:
page 3/ #6   page 6/ #7   page 10/ #4
page 13/ #8   page 20/ #8   page 24/ #6   page 28/ #3
page 33/ #1   page 44/ #11, 16   page 55/ #22, 24

1. A *perfect set* is a non-empty compact set such that every point of it is a limit point. Show that a perfect set is uncountable. (Hint: show that if a countable compact set is written as $\{x_1, x_2, \dots\}$, then $X_n = X - \{x_1, \dots, x_n\}$ has the same limit points as X for every $n \in \mathbb{N}$.)

2. Let (X, d) be a metric space and use $L(G)$ to denote the set of all bounded Lipschitz functions from X to $\mathbb{C}$. Define a metric on $L(G)$ by
$$\rho(f, g) = \|f - g\|_\infty + \inf\{M : M \text{ is a Lipschitz constant for } f - g\}.$$
Show that $(L(G), \rho)$ is a complete metric space.

3. Define the metric space X to be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$ with $d(f, g) = \|f - g\|$, and let K be the set of all $f \in X$ such that
$$\|f\|_\infty + \inf\{M : M \text{ is a Lipschitz constant for } f\} \le 1.$$
(a) Show that if (f_n) is a sequence of functions such that $\|f_n\|_\infty \le 1$ for all n, then there is a subsequence (f_{n_k}) such that the sequence $(f_{n_k}(x))$ converges for every rational x. (Hint: write the rationals as a sequence (q_m) and show that there is a subsequence that converges at $x = q_1$, then at $x = q_1$ and at $x = q_2$, etc.)
(b) Show that the subsequence (f_{n_k}) from part (b) converges uniformly if the functions f_n are all in K.
(c) Conclude that K is a compact subset of X.