SOLUTIONS TO PRACTICE FINAL EXAM

1. Define a new sequence \((b_n)\) by \(b_n = a_n + 1 - a_n\). Then \(\lim_{n \to \infty} (b_n + b_{n-1}) = 0\). Given \(\varepsilon > 0\), there is a natural number \(N\) such that, if \(n \geq N\), then \(|b_n + b_{n-1}| < \varepsilon/2\). We now prove by induction (on \(k\)) that

\[
|b_{N+k}| \leq k\frac{\varepsilon}{2} + |b_N|.
\]

For \(k = 1\), we have

\[
|b_{N+1}| = |b_n - (b_N + b_{N+1})| \leq |b_N| + |b_N + b_{N+1}| \leq \frac{\varepsilon}{2} + |b_N|.
\]

If the inequality is true for \(k = m\), then

\[
|b_{N+(m+1)}| = |b_{N+m} - (b_{N+m} + b_{N+(m+1)})| \leq |b_{N+m}| + |b_{N+m} + b_{N+(m+1)}| \\
\leq m\frac{\varepsilon}{2} + |b_N| + \frac{\varepsilon}{2} \leq (m + 1)\frac{\varepsilon}{2} + |b_N|.
\]

Now take \(M\) to be a natural number greater than \(N\) such that \(|b_N|/M < \varepsilon/2\). If \(n \geq M\), then

\[
\left|\frac{b_n}{n}\right| \leq \frac{n - N}{n} \frac{\varepsilon}{2} + \frac{|b_N|}{n} < \varepsilon.
\]

It follows that \(\lim_{n \to \infty} b_n/n = 0\).

2. For \(n = 1\), we have

\[
\sum_{k=1}^{1} F_k = F_1 = 1 = F_{1+2} - 1.
\]

If the statement is true for \(n = m\), then

\[
\sum_{k=1}^{m+1} F_k = \sum_{k=1}^{m} F_k + F_{m+1} = F_{m+2} - 1 + F_{m+1} = F_{m+3} - 1.
\]

3. In both parts, we will use the interval \([0, 1]\).

   (a) Use

   \[
   f(x) = \begin{cases} 
   \frac{1}{2} & \text{if } x = 0, \\
   x & \text{if } 0 < x < 1, \\
   \frac{1}{2} & \text{if } x = 1.
   \end{cases}
   \]

   (b) Use

   \[
   f(x) = \begin{cases} 
   1 & \text{if } x = 0, \\
   x & \text{if } 0 < x < 1, \\
   0 & \text{if } x = 1.
   \end{cases}
   \]
4. By algebra,
\[
\frac{a^2f(x+bh) - b^2f(x+ah) + (b^2 - a^2)f(x)}{(a^2b - b^2a)h} = \frac{a^2[f(x+bh) - f(x)] - b^2[f(x+ah) - f(x)]}{(a^2b - b^2a)h},
\]
so
\[
\lim_{h \to 0} \frac{a^2f(x+bh) - b^2f(x+ah) + (b^2 - a^2)f(x)}{(a^2b - b^2a)h} = \frac{a^2}{a^2 - ba} \lim_{h \to 0} \frac{f(x+bh) - f(x)}{bh} - \frac{b^2}{ab - b^2} \lim_{h \to 0} \frac{f(x+ah) - f(x)}{ah} = \left(\frac{a^2}{a^2 - ba} - \frac{b^2}{ab - b^2}\right)f'(x) = f'(x).
\]

5. The substitution \( u = e^x \) (so \( x = \ln u \) and \( dx = du/u \)) gives
\[
\int_0^\infty \sin(e^x) \, dx = \int_1^\infty \frac{\sin u}{u} \, du,
\]
which converges. (More formally, we would say that
\[
\int_0^\infty \sin(e^x) \, dx = \lim_{b \to \infty} \int_0^b \sin(e^x) \, dx = \lim_{b \to \infty} \int_1^{\ln b} \frac{\sin u}{u} \, du,
\]
which exists because the improper integral
\[
\int_1^\infty \frac{\sin u}{u} \, du
\]
converges.)

6. Because the sequence \((b_k)\) is bounded, there is a number \(M\) such that \(|b_k| \leq M\) for all \(k\). Because \(\sum |a_k|\) converges, given \(\varepsilon > 0\), there is a natural number \(N\) such that, if \(n \geq m \geq N\), then
\[
\sum_{k=m}^{n} |a_k| \leq \frac{\varepsilon}{M}.
\]
Now, if \(n \geq m \geq N\), then
\[
\sum_{k=m}^{n} |b_k a_k| \leq \sum_{k=m}^{n} M |a_k| = M \sum_{k=m}^{n} |a_k| < \varepsilon,
\]
which means that \(\sum a_k b_k\) converges absolutely.

7. This is an alternating series, so set \(a_k = 1/(k \ln k)\). Then \(a_k > 0\), \((a_k)\) is a decreasing sequence, and \(a_k \to 0\) as \(k \to \infty\). Therefore the series converges by the alternating series test.

8. FALSE. Take
\[
f(x) = \begin{cases} 
1 & \text{if } x = 0, \\
\frac{1}{x} & \text{if } x > 0,
\end{cases}
\]
and define \(x_n = 1/n\). Then \(\lim_{n \to \infty} f(x_n) = \infty\), but \(\lim_{x \to \infty} f(x) = 0\).
9. First, because the equation is a cubic, it has at least one root (by the intermediate value theorem). If we define \( f(x) = x^3 + ax + b \), then the mean value theorem tells us that if it has more than one root, then the equation \( f'(x) = 0 \) also has at least one root. However, \( f'(x) = 3x^2 + a \), which is never zero because \( a > 0 \). Therefore the equation has exactly one solution.

10. Take \( x_n = n\pi \) and \( t_n = x_n + 1/n \). Then \( |x_n - t_n| \to 0 \) as \( n \to \infty \) but

\[
|f(x_n) - f(t_n)| = t_n \left| \sin \left( \frac{1}{n} \right) \right|.
\]

Because \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \), it follows that there is a natural number \( N \) such that \( \sin(1/n) \geq 1/(2n) \) if \( n \geq N \). Hence, if \( n \geq N \), then

\[
|f(x_n) - f(t_n)| \geq \left( n\pi + \frac{1}{n} \right) \frac{1}{2n} \geq \frac{\pi}{2}.
\]

Therefore \( f \) is not uniformly continuous because a theorem in the book says that, if \( f \) is uniformly continuous on \( D \), then for any two sequences \( (x_n) \) and \( (t_n) \) of points in \( D \), with \( \lim_{n \to \infty} |x_n - t_n| = 0 \), we have \( \lim_{n \to \infty} f(x_n) - f(t_n) = 0 \).