HOMEWORK #5 SOLUTIONS

Section 2.2

8. (a) We will show that $A < B + \varepsilon$ for every $\varepsilon > 0$. So let $\varepsilon > 0$ be given. Then there are numbers $n_1$ and $n_2$ such that $|a_n - A| < \varepsilon/2$ if $n \geq n_1$ and $|b_n - B| < \varepsilon/2$ if $n \geq n_2$. We now choose $n = \max\{n_1, n_2\}$. Then

$$A - B = (A - a_n) + (b_n - B) + (a_n - b_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0 = \varepsilon,$$

so $A < B + \varepsilon$ as claimed.

(b) There are lots of examples. I like $a_n = 0$ and $b_n = \frac{1}{n}$ but there are many others.

11. (d) The limit exists and it’s zero. To see this, let $m \in \mathbb{N}$ be the number from Theorem 1.6.8 such that $m > |r|$, and let $\varepsilon > 0$ be given. Then Theorem 1.6.8 gives a number $n^\ast$ such that

$$n^\ast > \max\{\frac{1}{\varepsilon}, \frac{m}{m!}, m + 1\}.$$

If $n \geq n^\ast$, then

$$\left|\frac{r^n}{n!} - 0\right| = \left|\frac{r^n}{n!}\right| \leq \frac{m}{m} \frac{m}{m-1} \cdots \frac{m}{1} \frac{m}{m-1} \cdots \frac{m}{1}.$$

But $m/n \leq m/n^\ast$ so $m/s \leq 1$ if $s = m, \ldots, 1$, so

$$\left|\frac{r^n}{n!} - 0\right| \leq \frac{m}{m^\ast} \frac{m}{m!} < \varepsilon.$$

(i) Let $b_n = \sqrt{n}$ and $c_n = \sqrt{2n}$. Then $b_n \leq a_n \leq c_n$ and we know that $(b_n)$ converges to 1 by Exercise 15 of Section 2.1 while $(c_n)$ converges to 1 by combining Exercises 14 and 15 of Section 2.1 with Theorem 2.2.1(b). It then follows from the Squeeze Theorem that $(a_n)$ converges to 1.

13. It is not true that the sequence $(b_n)$ must converge. For example, we could have $a_n = 1/n$ and $b_n = \sqrt{n}$. Then $a_nb_n = 1/\sqrt{n}$, so $a_nb_n \to 0$, but $(b_n)$ diverges.

Section 2.3

4. (a) Take $b_n = a_n$.

(b) Take $b_n = (-1)^na_n$.

8. First, we rewrite $a_n = n^{p-q}b_n$ with

$$b_n = \frac{s_p + s_{p-1}n^{-1} + \cdots + a_0n^{-p}}{t_q + t_{q-1}n^{-1} + \cdots + t_0n^{-q}}.$$

From the limit theorems along with the equation $s_p t_q \neq 0$ (which is the same as $s_p \neq 0$ and $t_q \neq 0$), we have that $(b_n)$ converges to $s_p/t_q$, which is not zero. It then follows from
Theorem 2.1.12, that $b_n$ is bounded, so Theorem 2.3.3(b) tells us that $(a_n)$ diverges to infinity.

Section 2.4
1. One possibility is

$$a_n = \begin{cases} n & \text{if } n \text{ is even} \\ n/2 & \text{if } n \text{ is odd.} \end{cases}$$

6. (a) We can write

$$a_n = \frac{1}{2 + \frac{1/2}{2n + 1}},$$

so $(a_n)$ is decreasing, and it is bounded below by $1/2$. Therefore it converges by Exercise 5 of Section 2.4 (which is the “decreasing” version of Theorem 2.4.2).

(e) Now we have to be a little trickier. Let’s look at $a_{n+1}/a_n$:

$$\frac{a_{n+1}}{a_n} = \frac{3n+1}{1+3n+2} \cdot \frac{1+3n}{3n} = \frac{3+3n+3n+1}{1+3n+2} \leq \frac{3+3n+2}{1+3n+2} \leq 1,$$

so the sequence is decreasing. Since the terms are positive, it’s also bounded below, so it converges by Exercise 5.

(f) Now we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{3 \cdot \cdots \cdot (2n-1)}{1 \cdot 3 \cdot \cdots \cdot (2n+1)} = \frac{n+1}{2n+1} \leq 1,$$

so the sequence is decreasing. Again the terms are positive, so it’s bounded below and hence convergent.

7. (a) Now

$$a_{n+1} - a_n = \frac{1}{(n+1)^2} > 0,$$

so the sequence is increasing. Exercise 2(t) from Section 1.3 says that $a_n \leq 2 - 1/n$, so the sequence is also bounded. Therefore it converges by Theorem 2.4.2. In addition $a_n \geq 1$, so Theorem 2.2.1(f) says that $1 \leq A \leq \lim_{n \to \infty} (2 - 1/n)$. This last limit is 2. In addition, $a_n \geq 5/4$ if $n \geq 2$, so Theorem 2.2.1(f) also says that $A \geq 5/4$.

Section 2.5
3. (a) $a_n = (-1)^n n/(n+1)$ has exactly two accumulation points $-1$ and 1. (The problem didn’t ask you to prove that these are all the accumulation points, but here’s a proof that they are accumulation points anyway. First, given $\varepsilon > 0$, Theorem 1.6.8 says that there is an integer $n^*$ such that $n^* > 1/\varepsilon$. Then

$$|a_{2n^*} - 1| = \left| \frac{2n^*}{2n^*+1} - 1 \right| = \frac{1}{2n^*+1} \leq \frac{1}{n^*} < \varepsilon,$$

$a_{2n^*} \in S$, and $a_{2n^*} \neq 1$. Similarly $|a_{2n^*+1} - (-1)| < \varepsilon$, $a_{2n^*+1} \in S$, and $a_{2n^*+1} \neq -1$. Hence $-1$ and 1 are accumulation points. Showing that there aren’t any more accumulation points...
is similar but a lot of work!)
(b) Same as (a).
(c) $S = (0, 1) \cup \{2\}$.
(d) $S = \{-1\} \cup (0, 1) \cup \{2\}$.

4. (a) If $s_0$ is an accumulation point for $S$, then, for each positive integer $n$, there is a point $a_n \in S$ with $a_n \neq s_0$ and $|a_n - s_0| < 1/n$ (this is just the point corresponding to $\varepsilon = 1/n$). To see that $(a_n)$ converges to $s_0$, let $\varepsilon > 0$ be given. Then Theorem 1.6.8 gives a natural number $n^*$ such that $n^* > 1/\varepsilon$. If $n \geq n^*$, then

$$|a_n - s_0| < \frac{1}{n} \leq \frac{1}{n^*} < \varepsilon,$$

so $a_n \to s_0$.

Conversely, if there is a sequence $(a_n)$ in $S$ with $a_n \neq s_0$ for all $n$ and $a_n \to s_0$, then, given $\varepsilon > 0$, there is a natural number $n^*$ such that $|a_n - s_0| < \varepsilon$ if $n \geq n^*$. It follows that $a_{n^*} \in S$, $a_{n^*} \neq s_0$ and $|a_{n^*} - s_0| < \varepsilon$. Hence $s_0$ is an accumulation point of $S$.
(b) It’s necessary because the set $S = \{2\}$ has no accumulation points but the sequence $a_n = 2$ for all $n$ is a sequence of points in $S$ such that $a_n \to s_0$. 