Math 414
Professor Lieberman
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HOMEWORK #14 SOLUTIONS

Section 7.1

4. (a) FALSE. Take $a_k = 1/k$ and $b_k = -1/k$.
(b) FALSE. Take $a_k = 1/k$ and $b_k = 1/k$.
(c) TRUE. Let's notice that this statement has the form $(A\&B) \Longrightarrow C$, which is equivalent to $A \Longrightarrow (B \Longrightarrow C)$, which, in turn, is equivalent to $A \Longrightarrow (\,C \Longrightarrow\, B)$ or $(A\&\,C) \Longrightarrow\, B$. In words, this statement is "If $\sum a_k$ converges and $\sum (a_k + b_k)$ converges, then $\sum b_k$ converges. This last statement follows from Theorem 7.1.18.
(d) FALSE. Take $a_k = 0$ and $b_k = 1/k$.

11. Take $a_k = b_k = 1/2^k$. Then $A = B = 1$, but
$$\sum a_k b_k = \sum \frac{1}{4^k} = \frac{1}{1 - 1/4} = \frac{4}{3} \neq 1.$$

Section 7.2

1. (d) Recall (from page 99) that $(2k)!! = 2k \cdot (2k-2) \cdot \ldots \cdot 2$ and $(2k+1)!! = (2k+1)(2k-1) \cdot \ldots \cdot 3 \cdot 1$. (In other words, $(2k)!!$ is the product of all the even numbers up to and including $2k$ and $(2k+1)!!$ is the product of all odd numbers up to and including $2k+1$.) Therefore
$$\frac{(2k)!!}{(2k+1)!!} = \frac{1}{2k+1}\frac{2k}{2k-2}\cdots\frac{2}{1} \geq \frac{2}{2k+1} \geq \frac{1}{k+1}.$$
But $\sum \frac{1}{k+1}$ diverges. (This series is the harmonic series without its first term, which doesn't affect convergence.)
(g) If $k > 4$, then $\ln k \geq 2$, so $1/(\ln k)^k \leq 1/2^k$. Because $\sum 1/2^k$ is a geometric series with $r < 1$, it converges, so the original series converges by the comparison test.
(v) Set $b_k = (2/e)^k$. Then
$$\frac{a_k}{b_k} = \frac{2^k \sin(e^{-k})}{(2/e)^k} = \frac{\sin(e^{-k})}{e^{-k}},$$
so
$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sin(e^{-k})}{e^{-k}} = 1.$$
Again, $\sum b_k$ is a convergent geometric series, so it converges, and hence the original series converges by the limit comparison test.

2. (a) Since $f(x) = 1/(x(\ln x)^p)$ is a decreasing function for $x \geq 2$, we can use the integral test. The substitution $u = \ln x$ gives
$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_{\ln 2}^\infty \frac{1}{u^p}\,du,$$
and this second integral converges because $p > 1$. Therefore the first integral converges, so the series converges by the integral test.

5. Since $\lim\limits_{k\to\infty} a_k = 0$, there is a natural number N such that $a_n \le 1$ if $n \ge N$. Therefore $0 \le (a_n)^2 \le a_n$ for $n \ge N$, so the series converges by the limit test. (Or you can use the limit comparison test.)

7. (a) Given $\varepsilon > 0$, there is a natural number N such that
$$\sum_{k=N}^{\infty} a_k < \frac{\varepsilon}{2}.$$
If $n \ge 2N$, then
$$na_n \le 2(a_N + a_{N+1} + \cdots + a_n) < \varepsilon.$$
Therefore $\lim_{n\to\infty} na_n = 0$.
(b) Take
$$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is a perfect square,} \\ \frac{1}{n^2} & \text{otherwise.} \end{cases}$$
Then
$$\sum a_k \le 2\sum_{n=1}^{\infty} \frac{1}{n^2},$$
so $\sum a_k$ converges but $\lim\limits_{n\to\infty} na_n$ does not exist.
(c) From Exercise 7.2.2(a), the series $\sum 1/(\ln k)$ diverges but
$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

13. First, we have $(a_k + b_k)^2 \le 2(a_k)^2 + 2(b_k)^2$ and $|a_k b_k| \le (a_k)^2 + (b_k)^2$, so the series converge by the comparison test. Now Exercise 1.7.10(b) says that
$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left[\sum_{k=1}^{n} (a_k)^2\right]\left[\sum_{k=1}^{n} (b_k)^2\right],$$
and, taking the limit as $n \to \infty$ gives
$$\left(\sum a_k b_k\right)^2 \le \left[\sum (a_k)^2\right]\left[\sum (b_k)^2\right].$$
Similarly, Exercise 1.7.10(a) gives
$$\left[\sum_{k=1}^{n} (a_k)^2 + (b_k)^2\right]^{1/2} \le \left[\sum_{k=1}^{n} (a_k)^2\right]^{1/2} + \left[\sum_{k=1}^{n} (b_k)^2\right]^{1/2},$$
and, taking the limit as $n \to \infty$ gives
$$\left[\sum (a_k)^2 + (b_k)^2\right]^{1/2} \le \left[\sum (a_k)^2\right]^{1/2} + \left[\sum (b_k)^2\right]^{1/2}.$$

Section 7.3

4. (a) From Exercise 7.2.2a, the series diverges, but
$$\lim_{k\to\infty} \frac{(k+1)\ln(k+1)}{k\ln k} = 1.$$

5. (a) Use the sequence in Example 7.3.7.
(b) Take $b_k = e^k$.

7. (c) By the ratio test,

$$\frac{e^{k+1}\sin(2^{-k-1})}{e^k\sin(2^{-k})} = e\frac{\sin(2^{-k-1})}{\sin(2^{-k})} = \frac{e}{2}\frac{\sin(2^{-k-1})}{2^{-k-1}}\frac{\sin(2^{-k})}{2^{-k}}.$$

Since $\lim\limits_{h\to 0}\sin h/h = 1$, we see that

$$\lim_{k\to\infty}\frac{e^{k+1}\sin(2^{-k-1})}{e^k\sin(2^{-k})} = \frac{e}{2} > 1,$$

so the series DIVERGES.
(e) By the root test,

$$\sqrt[k]{\frac{3^{2k+1}}{k^{2k}}} = \sqrt[k]{3}\frac{3^2}{k^2} \leq \frac{27}{k^2} < \frac{1}{2}$$

for $k \geq 8$, so the series CONVERGES.
(f) By the root test,

$$\sqrt[k]{(\sqrt[k]{k}-1)^k} = \sqrt[k]{k}-1.$$

From Exercise 2.1.15, we have

$$\lim_{k\to\infty}\sqrt[k]{(\sqrt[k]{k}-1)^k} = 0,$$

so the series CONVERGES.

11. (a) If $\sqrt[n]{a_n} \leq \alpha$ for $n \geq N$, then $a_n \leq \alpha^n$ for $n \geq N$, so the series converges by the comparison test because $\sum \alpha^n$ is a convergent geometric series.
(b) If $\sqrt[n]{a_n} \geq 1$ for $n \geq N$, then $a_n \geq 1$ for $n \geq N$, so the series diverges by the comparison test because $\sum 1$ diverges.

Section 7.4

2. (c) Since $\lim\limits_{k\to\infty} e^k/k^4 = \infty$, the series DIVERGES by the nth term test.
(e) This is an alternating series and the sequence (a_k), defined by

$$a_k = \frac{1}{k\sqrt[k]{k}}$$

is eventually decreasing by Exercise 5.2.1(g). In addition $0 \leq a_k \leq 1/k$, so $\lim\limits_{k\to\infty} a_k = 0$, so the series CONVERGES.

3. First, we calculate a derivative:

$$\frac{d}{dx}\left(\frac{(\ln x)^p}{x}\right) = \frac{p(\ln x)^{p-1}-(\ln x)^p}{x^2},$$

and this derivative is negative if $x > e^p$. Therefore the sequence (a_k) defined by

$$a_k = \frac{(\ln k)^p}{k}$$

is eventually decreasing. It's easy to see that $\lim_{k\to\infty} a_k = 0$, so the series converges for any positive p. (In particular for any $p \in \mathbb{N}$.)

6. (a) We have
$$\frac{\arctan k}{k^2} \le \frac{\pi/2}{k^2},$$
and $\arctan k$ is positive for all k, so the series converges absolutely.
(c) It's easy to check that $1/(k \ln k)$ is decreasing and that $\lim_{k\to\infty} 1/(k\ln k) = 0$, so the series converges. However, we know that $\sum 1/(k\ln k)$ diverges, so the original series converges conditionally.

Section 7.5

1. FALSE. Take $a_k = 1$ for all k.

6. FALSE. Take $a_k = 1/k$. Then $\sum (a_k)^2$ is just a p-series with $p = 2$ so it converges, but $\sum a_k$ is the harmonic series which diverges.

15. TRUE. It's an alternating series wit $a_k = 1/\sqrt[p]{k}$. Clearly (a_k) is decreasing and $\lim_{k\to\infty} = 0$, so the series converges.