Section 6.4

5. (c) Integrate by parts (with \( u = \sin(\ln x) \) and \( dv = dx \)) to obtain

\[
\int \sin(\ln x) \, dx = x \sin(\ln x) - \int \frac{x \cos(\ln x)}{x} \, dx = x \sin(\ln x) - \int \cos(\ln x) \, dx.
\]

Another integration by parts (with \( u = \cos(\ln x) \) and \( dv = dx \)) then gives

\[
\int \cos(\ln x) \, dx = x \cos(\ln x) - \int \frac{x - \sin(\ln x)}{x} \, dx = x \sin(\ln x) + \int \sin(\ln x) \, dx.
\]

Therefore

\[
\int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx,
\]

so

\[
\int \sin(\ln x) \, dx = \frac{1}{2} [x \sin(\ln x) - x \cos(\ln x)].
\]

(i) The integrand \( x^{-2} \) is undefined (and unbounded) at \( x = 0 \), so the integral does not exist.

7. (c) \( F'(x) = 2x \arctan(x^4) - 2 \arctan(4x^2) \).

8. We have

\[
F(x) - F(t) = \int_t^x f(s) \, ds
\]

for \( a \leq x < t \leq b \), and there are numbers \( m \) and \( M \) such that \( m \leq f(s) \leq M \) for \( s \in [a, b] \).

It follows from Theorem 6.3.2 that

\[
m(t - x) = \int_t^x m \, ds \leq \int_t^x f(s) \, ds \leq \int_t^x M \, ds \leq M(t - x),
\]

and therefore

\[
|F(x) - F(t)| \leq \max\{|m|, |M|\}|x - t|,
\]

so \( F \) is Lipschitzian.

10. (d) If we rewrite

\[
n^{-3/2} \sqrt{k} = \frac{1}{n} \sqrt{\frac{k}{n}},
\]

then Exercise 6.2.10 gives

\[
\lim_{n \to \infty} n^{-3/2} \sum_{k=1}^{n} \sqrt{k} = \int_0^1 \sqrt{x} \, dx = \frac{2}{3}.
\]
(f) Now we have
\[ \frac{n}{n^2 + k^2} = \frac{1}{n} \left( \frac{1}{n} + \left( \frac{k}{n} \right)^2 \right), \]
so
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2} = \int_0^1 \frac{1}{1 + x^2} \, dx = \arctan x \bigg|_0^1 = \frac{\pi}{4}.
\]

22. To make the formulas easier to read, define
\[ F(x) = \int_0^x f(t) \, dt. \]
Now we integrate by parts with \( u = [f(x)]^2 \) and \( dv = f(x) \, dx \) (so \( v = F(x) \)) to obtain
\[
\int_0^1 [f(x)]^3 \, dx = [f(x)]^2 F(x) \bigg|_0^1 - \int_0^1 2f(x)f'(x)F(x) \, dx
= [f(1)]^2 F(1) - \int_0^1 2f(x)f'(x)F(x) \, dx
= [f(1)]^2 F(1) - \int_0^1 2f(x)f'(x)[F(1) - F(x)] \, dx.
\]
Now we use the hypothesis \( 0 < f' \) to conclude that \( f \geq 0 \) (because \( f \) is increasing and \( f(0) = 0 \)). It also follows that \( F \geq 0 \) by similar reasoning. Therefore \( f(x)f'(x)[F(1) - F(x)] \leq f(x)[F(1) - F(x)] \) because \( f' \leq 1 \). From our previous calculation, we conclude that
\[
\int_0^1 [f(x)]^3 \, dx \leq \int_0^1 2f(x)[F(1) - F(x)] \, dx = \int_0^1 2f(x)F(1) - 2f(x)F(x) \, dx
= 2F(x)F(1) - [F(x)]^2 \bigg|_0^1 = F(1)^2 = \left[ \int_0^1 f(x) \, dx \right]^2.
\]

Section 6.5

3. (a) \( p > 1 \) because
\[
\int_1^b \frac{1}{x^p} \, dx = \left[ \frac{1}{-(p+1)x^{p+1}} \right]_1^b = p - 1 - (p - 1)b^{1-p},
\]
and this expression has a limit as \( b \to \infty \) if \( p > 1 \).
(b) \( p < 1 \) because
\[
\int_1^b \frac{1}{x^p} \, dx = p - 1 - (p - 1)b^{1-p},
\]
and this expression has a limit as \( b \to 0 \) if \( p < 1 \).

7. (a) The Cauchy principal value for \( \int_{-1}^1 (1/x) \, dx \) is
\[
\lim_{\epsilon \to 0^+} \left( \int_{-\epsilon}^{\epsilon} \frac{1}{x} \, dx + \int_{\epsilon}^{1} \frac{1}{x} \, dx \right) = \lim_{\epsilon \to 0^+} (\ln \epsilon - \ln \epsilon) = 0,
\]
but
\[
\int_0^1 \frac{1}{x} \, dx = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} \frac{1}{x} \, dx = \lim_{\epsilon \to 0^+} -\ln \epsilon = \infty,
\]
so $1/x$ is not improper Riemann integrable on $[-1, 1]$.

(b) The Cauchy principal value for $\int_{-1}^{1} (1/x) \, dx$ is

$$\lim_{\varepsilon \to 0^+} \left( \int_{-1}^{-\varepsilon} \frac{1}{|x|} \, dx + \int_{\varepsilon}^{1} \frac{1}{|x|} \, dx \right) = \lim_{\varepsilon \to 0^+} (-\ln \varepsilon - \ln \varepsilon) = \infty.$$  

(c) The Cauchy principal value is

$$\lim_{\varepsilon \to 0^+} \int_{-1}^{1-\varepsilon} \frac{4}{x-1} \, dx + \int_{1+\varepsilon}^{2} \frac{4}{x-1} \, dx = \lim_{\varepsilon \to 0^+} (4 \ln \varepsilon - 4 \ln 2 - [4 \ln 1 - 4 \ln \varepsilon]) = 4 \ln 2.$$  

9. (i) First, notice that $\lim_{x \to \infty} |\ln x/(1+x^{1/2})| = 0$, so $|\ln x/(1+x^2)| \leq x^{-3/2}$ for $x$ sufficiently large and therefore

$$\int_{1}^{\infty} \frac{\ln x}{1+x^2} \, dx$$  

converges. Also,

$$\int_{k}^{1} \frac{\ln x}{1+x^2} \, dx = \int_{1/k}^{1} \frac{-\ln t}{1+((1/t)^2)} \, dt$$  

(using the change of variables $t = 1/x$), so

$$\int_{k}^{1} \frac{\ln x}{1+x^2} \, dx = -\int_{1}^{1/k} \frac{\ln x}{1+x^2} \, dx.$$  

Therefore

$$\int_{0}^{1} \frac{\ln x}{1+x^2} \, dx$$  

also converges and

$$\int_{1}^{\infty} \frac{\ln x}{1+x^2} \, dx = -\int_{0}^{1} \frac{\ln x}{1+x^2} \, dx.$$  

It follows that

$$\int_{0}^{\infty} \frac{\ln x}{1+x^2} \, dx = 0.$$  

19. If $0 < x < y$, then integration by parts with $u = 1/t$ and $dv = \sin t \, dx$ (so $v = 1 - \cos t$) gives

$$\int_{x}^{y} \frac{\sin t}{t} \, dt = \frac{1 - \cos y}{y} \bigg|_{x}^{y} + \int_{x}^{y} \frac{1 - \cos t}{t^2} \, dt.$$  

Note that $(1 - \cos x)/x \to 0$ as $x \to 0^+$, and $(1 - \cos y)/y \to 0$ as $y \to \infty$. In addition, $|(1 - \cos t)/t^2| \leq 2/(1+t^2)$, so the integral

$$\int_{x}^{y} \frac{1 - \cos t}{t^2} \, dt$$  

converges as $x \to 0^+$ and $y \to \infty$. Therefore

$$\int_{0}^{\infty} \frac{\sin x}{x} \, dx$$  

converges and

$$\int_{0}^{\infty} \frac{\sin x}{x} \, dx = \int_{0}^{\infty} \frac{1 - \cos x}{x^2} \, dx.$$
To see that the integral is only conditionally convergent, we first observe that
\[
\int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} \, dx \geq \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| \, dx = \frac{2}{(n+1)\pi}.
\]
It follows that
\[
\int_{0}^{n\pi} \frac{|\sin x|}{x} \, dx \geq \sum_{k=1}^{n} \frac{2}{k\pi}.
\]
This sum diverges to infinity as \( n \to \infty \), so the integral diverges.

22. Since \( f \) is continuous on \([0, \infty)\) and \( f(x) \leq 1/x^2 \), the integral is convergent. Now we use Exercise 19 to see that
\[
\int_{0}^{\infty} \frac{\sin x}{x} \, dx = \int_{0}^{\infty} \frac{1 - \cos x}{x^2} \, dx = 2 \int_{0}^{\infty} \frac{\sin^2(x/2)}{x^2} \, dx
\]
from the trig identity \( 1 - \cos x = \sin^2(x/2) \). Now the change of variables \( u = x/2 \) yields
\[
\int_{0}^{\infty} \frac{\sin^2(x/2)}{x^2} \, dx = \int_{0}^{\infty} \frac{\sin^2 u}{4u^2} \, 2du = \frac{1}{2} \int_{0}^{\infty} \frac{\sin^2 u}{u^2} \, dx.
\]

Section 6.7

1. TRUE. \( \int_{0}^{\infty} \sin x \, dx \) diverges but \( \int_{0}^{b} \sin x \, dx \) is between 0 and 2 for any positive \( b \).

8. TRUE. (This is similar to the argument for Dirichlet’s function.) First notice that \( L(P, f) = 0 \) for any partition \( P \). Now let \( q \) be an integer greater than \( 1 \) and set \( n = q^2 - q + 1 \). If \( P \) is the partition \( \{x_0, \ldots, x_n\} \) with \( x_1 = 1/q \) and \( x_i = (q+i-1)/q^2 \) for \( i = 2, \ldots, n \), then the maximum of \( f \) is 1 on \([x_0, x_1]\) and also on at most \( 2q \) intervals \([x_{i-1}, x_i]\) with \( i \geq 2 \). Therefore
\[
U(P, f) \leq \frac{1}{q} + 2q \left( \frac{1}{n} \right) = \frac{3q + 1}{q^2 - q + 1}.
\]
Now we note that \( \lim_{q \to \infty} (3q + 1)/(q^2 - q + 1) = 0 \) to conclude that we can make \( U(P, f) \) less than any given \( \varepsilon > 0 \) by taking \( q \) large enough.