HOMEWORK #11 SOLUTIONS

Section 5.5

14. FALSE. \( f(x) = x^3 - x \) has an inflection point at \( x = 0 \) because the second derivative switches from negative to positive there, but \( f'(0) = -1 \).

21. TRUE. \( f'(x) = 3x^2 + 8x - 1 \), which is zero when \( x = (-4 \pm \sqrt{19})/3 \), and neither one of these numbers is in the interval.

24. (b) FALSE. Take \( f(x) = x^3 - 3x \). Then the tangent line to this curve at \( x = 1 \) has zero slope, so it’s \( y = -2 \), which intersects the graph also at \( x = -2 \).

Section 5.6

11. (a) \( f(x) = x^{1/3} \) on \([-1, 1]\).
(b) \( f(x) = |x| \) on \([-1, 1]\).
(c) \( f(x) = 1/x \) on \((0, 1)\).

19. (a) Suppose first that \( f \) is increasing. Then, for any partition \( P \),

\[
\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = f(b) - f(a),
\]

so \( V_f(a, b) = f(b) - f(a) \) and \( f \) is of bounded variation.

If \( f \) is decreasing, then \( V_f(a, b) = f(a) - f(b) \) by similar reasoning so \( f \) is also of bounded variation in this case.

Because \( x^{1/3} \) is increasing on any interval, it is also of bounded variation on any bounded interval.

(h) \( f(x) = x^{1/3} \) on the interval \([-1, 1]\). (Strictly speaking, this example isn’t correct because \( f'(0) \) is infinite. To fix this situation, look at

\[
f(x) = \begin{cases} 
x^{3/2} \sin \frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]

on \([0, 1]\). Then

\[
f'(x) = \begin{cases} 
\frac{3}{2} x^{1/2} \sin \frac{1}{x} - x^{-1/2} \cos \frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]
which is unbounded because $f'(x_n) \to \infty$ for $x_n = 1/(n\pi)$. To prove that $f$ is of bounded variation requires some integration theory. Given a partition $P$, we have

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = |f(x_1)| + \sum_{k=2}^{n} \left| \int_{x_{k-1}}^{x_k} f'(x) \, dx \right|$$

$$\leq |f(x_1)| + \sum_{k=2}^{n} \int_{x_{k-1}}^{x_k} |f'(x)| \, dx \leq |f(x_1)| + \sum_{k=2}^{n} \int_{x_{k-1}}^{x_k} \frac{5}{2} x^{-1/2} \, dx$$

$$= |f(x_1)| + \frac{5}{2} \int_{x_2}^{1} x^{-1/2} \, dx = |f(x_1)| + \frac{5}{3} \left( 1 - (x_2)^{1/2} \right) \leq \frac{8}{3}.$$  

Because of this technical point, full credit will be awarded for saying $f(x) = x^{1/3}.$

21. (f) Let $L$ be a Lipschitz constant for $f$ and let $\varepsilon > 0$ be given. We then choose $\delta = \varepsilon/L.$ If $(a_k, b_k) \ (k = 1, \ldots, n)$ are $n$ nonoverlapping intervals with $\sum_{k=1}^{n} (b_k - a_k) < \delta$, then

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} L(b_k - a_k) = L \sum_{k=1}^{n} (b_k - a_k) < L\delta = \varepsilon.$$

24. (b) $f(x) = |x|$.

(e) Take $t \in (0, 1)$ so that $b = (1 - t)a + tc$. Then $t = (b - a)/(c - a)$ and $f(b) \leq (1 - t)f(a) + tf(c)$, so

$$f(b) - f(a) \leq tf(c) - f(a) = \frac{b - a}{c - a} [f(c) - f(a)],$$

and therefore

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a}.$$  

In addition, $1 - t = (c - b)(c - a)$, so

$$f(b) - f(c) \leq (1 - t)[f(a) - f(c)] = \frac{c - b}{c - a} [f(a) - f(c)],$$

so

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b}.$$  

(g) From part (f), $f'_+(x)$ is finite, so $\lim_{h \to 0^+} f(x + h) = f(x)$. (Just as in Theorem 5.1.6.) Similarly, $\lim_{h \to 0^-} f(x + h) = f(x)$, and therefore $\lim_{h \to 0} f(x + h) = f(x)$, which means that $f$ is continuous.

Section 6.1

7. Take $f$ to be the function from Example 6.1.8, and set $g = -f$. Then

$$\int_{a}^{b} f + g = \int_{a}^{b} 0 = 0, \quad \int_{a}^{b} f = 20, \quad \int_{a}^{b} g = 0.$$
Section 6.2
7. Since \( f \) is continuous on the closed, bounded interval \([a, b]\), it attains its maximum value at some point \( c \in [a, b] \). Suppose \( f(c) > 0 \), say \( f(c) = K \). Then there is a number \( \delta > 0 \) such that \( f(x) \geq K/2 \) for \( |x - c| < \delta \). Let \( \varepsilon > 0 \) be given and take \( P \) to be a partition such that
\[
\left| \int_a^b f - U(P, f) \right| < \frac{\varepsilon}{2},
\]
and let \( Q \) be the partition \( P \cup \{\max\{a, c - \delta\}, \min\{b, c + \delta\}\} \).

8. First notice that \( L(P, f) = 0 \) for any partition \( P \). Therefore, we only have to show that we can make \( U(P, f) \) small by choosing the partition \( P \) appropriately. Specifically, given \( \varepsilon > 0 \), chose \( q \in \mathbb{N} \) so that \( 1/q < \varepsilon/2 \) and then \( n \in \mathbb{N} \) so that \( q(q + 1)/n < \varepsilon/2 \). If \( P \) is the partition \( \{x_0, \ldots, x_n\} \) with \( x_i = i/n \), then there are at most \( q(q + 1)/2 \) subintervals in the partition on which the maximum of \( f \) is greater than \( 1/q \), and the maximum on each of these subintervals is at most \( 1/2 \). Therefore
\[
U(P, f) \leq \frac{q(q + 1)}{2} \left( \frac{1}{n} \right)^2 + \frac{1}{q}
\]
because the sum of the lengths of the subintervals on which the maximum of \( f \) is less than or equal to \( 1/q \) is less than or equal to \( 1 \). Therefore \( U(P, f) < \varepsilon \). It follows from Theorem 6.2.1 that \( f \) is integrable and, because the lower integral is zero, \( \int_0^1 f = 0 \). This does not contradict Exercise 7 because \( f \) is discontinuous.

Section 6.3
1 (a) NO. Take \( f(x) = 0 \) and \( g(x) = x \) on the interval \([-1, 1]\).

4. (a) First, notice that, for any \( y \) and \( z \) in \([a, b]\), we have
\[
f(y)g(y) - f(z)g(z) = f(y)[g(y) - g(z)] + g(z)[f(y) - f(z)]
\leq \sup |f||g(y) - g(z)| + \sup |g||f(y) - f(z)|.
\]
Now fix a partition \( P = \{x_0, \ldots, x_n\} \). If \( y \) and \( z \) are both in \([x_{k-1}, x_k]\), then \(|g(y) - g(z)| \leq M_k(g) - m_k(g)\) and \(|f(y) - f(z)| \leq M_k(f) - m_k(f)\). Now fix \( k \) and let \((y_n)\) be a sequence in \([x_{k-1}, x_k]\) such that \( f(y_n)g(y_n) \to M_k(fg) \) and let \((z_n)\) be a sequence in \([x_{k-1}, x_k]\) such that \( f(z_n)g(z_n) \to m_k(fg) \). Then
\[
M_k(fg) - m_k(fg) \leq \sup |f||M_k(g) - m_k(g)| + \sup |g||M_k(f) - m_k(f)|.
\]
After multiplying this inequality by \( x_k - x_{k-1} \) and summing on \( k \), it follows that
\[
U(P, fg) - L(P, fg) \leq \sup |f||U(P, g) - L(p, g)| + \sup |g||U(P, f) - L(P, f)|.
\]

To finish, we suppose that \( \varepsilon > 0 \) is given and choose a partition \( Q \) such that \( U(Q, f) - L(Q, f) < \varepsilon/(2\sup |g|) \) and a partition \( R \) such that \( U(R, g) - L(R, g) < \varepsilon/(2\sup |f|) \). If we then take \( P = Q \cup R \), then we see that \( U(P, fg) - L(P, fg) < \varepsilon \), so \( fg \) is integrable.

5. (b) Define \( f \) on \([0, 1]\) by
\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is rational}, \\
-1 & \text{if } x \text{ is irrational}.
\end{cases}
\]
then $f$ is not Riemann integrable for the same reason that the function in Example 6.1.8 isn’t. But $|f|(x) = 1$ for any $x \in [0, 1]$, so $|f|$ is continuous and therefore integrable.

13. For any real numbers $\alpha$ and $\beta$, we have

$$0 \leq \int_a^b (\alpha f + \beta g)^2,$$

so

$$-\alpha \beta \int_a^b fg \leq \frac{\alpha^2}{2} \int_a^b f^2 + \frac{\beta^2}{2} \int_a^b g^2.$$

Suppose first that $\int_a^b g^2 = 0$. Then take $\beta = -\int_a^b fg$ and suppose $\alpha > 0$. Then we can divide by $\alpha$:

$$\left(\int_a^b fg\right)^2 \leq \frac{\alpha}{2} \int_a^b f^2$$

for any $\alpha > 0$. Taking the limit as $\alpha \to 0$ then gives

$$\left(\int_a^b fg\right)^2 \leq 0 = \left(\int_a^b f^2\right) \left(\int_a^b g^2\right).$$

If $\int_a^b g^2$ isn’t zero, then take

$$\alpha = \int_a^b g^2, \quad \beta = -\int_a^b fg$$

to see that

$$\left(\int_a^b f^2\right) \left(\int_a^b g^2\right) \leq \frac{1}{2} \left(\int_a^b g^2\right)^2 \left(\int_a^b f^2\right) + \left(\int_a^b fg\right)^2 \left(\int_a^b g^2\right).$$

Therefore

$$\left(\int_a^b g^2\right) \left(\int_a^b fg\right)^2 \leq \left(\int_a^b g^2\right)^2 \left(\int_a^b f^2\right).$$

Dividing both sides of this equation by $\int_a^b g^2$ gives

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right) \left(\int_a^b g^2\right).$$