Math 307, Section C2
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FINAL EXAM

Directions: You only need to work any 6 out of the 8 problems on this exam. Each question is worth 16 points and you get 4 points for writing your name on the exam. To receive full credit, you must show all work. You may use a calculator to do the arithmetic, but you must show all steps in the calculations.

1. Suppose that the matrix A has the form
$$A = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$$
with positive entries a, b, c, and d. If A has three distinct real eigenvalues, how many of the eigenvalues are positive and how many are negative?

2. Find a basis for the subspace of $\mathbb{R}^5$ spanned by the vectors
$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 9 \\ 1 \\ 5 \end{bmatrix}.$$

3. Find all linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that
$$T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4. If
$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix},$$
find a matrix B so that $\ker(A) = \operatorname{im}(B)$.

5. Decide whether the following statements are true or false and justify your answer.
(a) If the quadratic form $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ is positive definite and A is a symmetric matrix, then all entries of A are positive.
(b) If A is a symmetric matrix and all entries of A are positive, then the quadratic form $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ is positive definite.

6. Perform Gram-Schmidt orthogonalization on the set of vectors
$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}.$$

7. Find a polynomial $f(t)$ of degree 3 so that $f(1) = 1$, $f(2) = 5$, $f'(1) = 2$, and $f'(2) = 9$. (Recall that f' is the derivative of f.)

8. According to fact 7.26 if A and B are similar matrices, then they have the same characteristic polynomial, the same determinant, the same trace, the same eigenvalues with the same algebraic and geometric multiplicities, and the same rank. On the other hand, two matrices can have all these properties the same without being similar. In this problem, we are going to examine the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Show that these matrices have the same characteristic polynomial, the same determinant, the same trace, the same eigenvalues with the same algebraic and geometric multiplicities, and the same rank.
(b) Show that any matrix

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix}$$

such that $SA = BS$ must have $s_{12} = s_{22} = s_{32} = s_{42} = 0$.
(c) Explain (using part (b)) why A and B are not similar.