

SOLUTIONS TO PRACTICE SECOND EXAM

1. (a) We check:

$$T((f_1 + f_2)(t)) = \int_{-2}^3 (f_1 + f_2)(t) dt = \int_{-2}^3 f_1(t) dt + \int_{-2}^3 f_2(t) dt = T(f_1(t)) + T(f_2(t)),$$

and

$$T(kf_1(t)) = \int_{-2}^3 (kf_1)(t) dt = k \int_{-2}^3 f_1(t) dt = kT(f_1(t)),$$

so it IS a linear transformation.

(b) If $f(t) = a + bt + ct^2$, then $T(f(t)) = bt + 2ct^2$. Therefore the image is the set of all polynomials of the form $f_1t + f_2t^2$, and the rank is 2. The kernel is the set of all constant polynomials, and the nullity is 1.

2. First, we write $0 = 0 + 0$ and recall that 0 is in V and in W . Next, suppose we have two elements $v_1 + w_1$ and $v_2 + w_2$ of $V + W$. Then

$$(v_1 + w_1) + (v_2 + w_2) = (v_1 + v_2) + (w_1 + w_2).$$

Because V and W are subspaces, $v_1 + v_2$ is in V and $w_1 + w_2$ is in W , so $(v_1 + w_1) + (v_2 + w_2)$ is in $V + W$. Finally, if $v + w$ is in $V + W$ and if k is a scalar, then

$$k(v + w) = kv + kw.$$

Again, kv is in V and kw is in W , so $k(v + w)$ is in $V + W$. Therefore $V + W$ is necessarily a subspace of \mathbb{R}^n .

3. First, we compute the reduced row echelon form of A .

$$\begin{aligned} & \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \end{bmatrix} \begin{array}{l} +(I) \\ -(I) \end{array} \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \div(-1) \\ & \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \begin{array}{l} +(II) \\ +(II) \end{array} \begin{bmatrix} 1 & -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \div(-1) \\ & \begin{bmatrix} 1 & -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} +2(III) \\ +3(III) \end{array} \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

This matrix tells us that the first, third and fifth variables are leading, so we can choose the second and fourth arbitrarily. Therefore a basis for the kernel is

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

and the kernel has dimension 2.

A basis for the image consists of the first, third and fifth columns, so it's

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\},$$

and the image has dimension 3.

4. We know that $\dim(\text{im}(B)) + \dim(\text{ker}(B))$ must be 3. If the kernel and image are equal, they have the same dimension, so this sum must be even, and therefore it can't equal three. So there is no 3×3 matrix B so that $\text{im}(B) = \text{ker}(B)$.