Math 307, Section C2
Professor Lieberman
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SECOND IN-CLASS EXAM

Directions: To receive full credit, you must show all work. You may use a calculator to
do the arithmetic, but you must show all steps in the calculations.

1. (25 points) Find an orthonormal basis for
\[
\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 2 \\ 13 \end{bmatrix} \right\}.
\]

2. (20 points) Consider two subspaces \( V \) and \( W \) of \( \mathbb{R}^n \). Let \( V + W \) be the set of all vectors in \( \mathbb{R}^n \) of the form \( \vec{v} + \vec{w} \), where \( \vec{v} \) is in \( V \) and \( \vec{w} \) is in \( W \). Is \( V + W \) necessarily a subspace of \( \mathbb{R}^n \)?

3. (30 points) Let \( A \) be the matrix
\[
A = \begin{bmatrix}
1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 0 & -2 & 2 \\
1 & -1 & -2 & 0 & 3
\end{bmatrix}.
\]
Find a basis for the kernel of \( A \), a basis for the image of \( A \) and determine their dimensions.

4. (15 points) If \( A \) is an orthogonal \( n \times n \) matrix, is \( A^2 \) necessarily orthogonal? If the answer is yes, explain why. If the answer is no, give an example.

5. (10 points) Is there a \( 3 \times 3 \) matrix \( B \) so that \( \text{im}(B) = \text{ker}(B) \)? Explain.
THIRD IN-CLASS EXAM

Directions: To receive full credit, you must show all work. (For problem 2(b), you do not need to calculate the eigenvalues.) You may use a calculator to do the arithmetic, but you must show all steps in the calculations.

1. (30 points) Compute the determinant of the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 1 & 3 & 3 \\
1 & 1 & 1 & 4
\end{bmatrix}
\]

2. (a) (15 points) Find all (complex) eigenvalues for the matrix

\[
\begin{bmatrix}
1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

(b) (15 points) Given that the matrix

\[
A = \begin{bmatrix}
1 & -3 & 3 \\
3 & -5 & 3 \\
6 & -6 & 4
\end{bmatrix}
\]

has eigenvalues \( \lambda = -2 \) and \( \lambda = 4 \) (and no other eigenvalues), find a basis for the eigenspaces of \( A \). Is there an eigenbasis?

3. (a) (15 points) Determine the characteristic polynomial of

\[
A = \begin{bmatrix}
a & b & c \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

in terms of \( a \), \( b \), and \( c \).

(b) (5 points) Use your answer to part (a) to find a \( 3 \times 3 \) matrix with characteristic polynomial \( \lambda^3 - 17\lambda^2 + 5\lambda - \pi \).

4. (10 points per part) An \( n \times n \) matrix \( A \) is called *skew-symmetric* if \( A^T = -A \).
   (a) Give an example of a nonzero \( 2 \times 2 \) skew-symmetric matrix.
   (b) Show that a skew-symmetric \( 3 \times 3 \) matrix \( A \) must have zero determinant. What does this say about the invertibility of \( A \)?