SOLUTIONS FOR PRACTICE FINAL

Corrected on May 9.

1. (Not for Spring, 2002)

2. First, we solve the homogeneous equation
\[ y'' - 2y' - 3y = 0. \]
The solution has the form \( y_h = e^{rt} \) with \( r^2 - 2r - 3 = 0 \) so \( r = 3, -1 \). Then the solution of the original equation has the form
\[ y = u_1 e^{3t} + u_2 e^{-t} \]
for functions \( u_1 \) and \( u_2 \) which solve the system of equations
\[
\begin{align*}
u_1' e^{3t} + u_2' e^{-t} &= 0 \\
u_1' 3e^{3t} + u_2' (-1)e^{-t} &= 3e^{2t}.
\end{align*}
\]Add these two equations to see that \( 4u_1' e^{3t} = 3e^{2t} \) or \( u_1' = (3/4)e^{-t} \). Plugging into the first equation for \( u_1' \) and \( u_2' \) gives \( u_2' = (-3/4)u_1' \), so
\[ y = -\frac{3}{4} e^{2t} + c_1 e^{3t} - \frac{1}{4} e^{2t} + c_2 e^{-t}. \]

3. This equation is exact, so the solution has the form \( \psi(x, y) = C \) with
\[
\psi(x, y) = \int (2xy^2 + 2y) \, dx = x^2 y^2 + 2xy + f(y),
\]
and
\[
\psi(x, y) = \int (2x^2 y + 2x) \, dy = x^2 y^2 + 2xy + g(x).
\]
If we set these two expressions equal, we see that \( f(y) = g(x) \), so \( f \) and \( g \) are constants, which we can take to be zero. Therefore \( \psi(x, y) = x^2 y^2 + 2xy \), and the solution of the differential equation is
\[ x^2 y^2 + 2xy = C. \]

4. The integrating factor \( \mu \) solves the equation
\[ \frac{\mu'}{\mu} = \cot x, \]
so \( \ln \mu = \ln(\sin x) \), or \( \mu = \sin x \). Therefore,
\[ ((\sin x)y)' = 2 \csc x \sin x = 2. \]
Integrating this equation gives \((\sin x)y = 2x + C\). From the initial condition \((\sin \frac{x}{2})1 = 2\frac{x}{2} + C\), which simplifies to \(C = 1 - \pi\). Therefore
\[
y = \frac{2x + 1 - \pi}{\sin x}.
\]

5. The differential equation has as its solution \(y = e^{rx}\) with \(r\) satisfying the equation \(r^2 + 4r + 5 = 0\). Therefore \(r = -2 \pm i\), and the general solution is
\[
y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x.
\]
Since
\[
y' = -2c_1 e^{-2x} \cos x - c_1 e^{-2x} \sin x - 2c_2 e^{-2x} \sin x + c_2 e^{-2x} \cos x,
\]
the initial conditions become \(c_1 = 1\) and \(-2c_1 + c_2 = 0\), so \(c_2 = 2\) and the solution is
\[
y = e^{-2x} \cos x + 2e^{-2x} \sin x.
\]

6. (a) First, we rewrite \(f\):
\[
f(t) = u_{\pi/2}(t) \sin t = u_{\pi/2}(t) [\cos(t - \frac{\pi}{2})],
\]
so
\[
\mathcal{L}\{f(t)\} = e^{-\pi s/2} \frac{s}{s^2 + 1}.
\]
Then the Laplace transform \(Y\) of the solution of the differential equation satisfies the algebraic equation
\[
(s^2Y(s) - s) + 2(sY(s) - 1) + Y(s) = e^{-\pi s/2} \frac{s}{s^2 + 1}
\]
and therefore
\[
Y(s) = \frac{s + 2}{s^2 + 2s + 1} + e^{-\pi s/2} \frac{s}{(s^2 + 2s + 1)(s^2 + 1)}.
\]
Since \(s^2 + 2s + 1 = (s + 1)^2\), we have
\[
\frac{s + 2}{s^2 + 2s + 1} = \frac{1}{s + 1} + \frac{1}{(s + 1)^2},
\]
and
\[
\frac{s}{(s^2 + 2s + 1)(s^2 + 1)} = \frac{A}{(s + 1)^2} + \frac{B}{s + 1} + \frac{C}{s^2 + 1} + \frac{Ds}{s^2 + 1}.
\]
The coefficients \(A, B, C, D\) satisfy the equation
\[
s = A(s^2 + 1) + B(s + 1)(s^2 + 1) + C(s + 1)^2 + Ds(s + 1)^2,
\]
or
\[
s = (B + D)s^3 + (A + B + C + 2D)s^2 + (B + 2C + D)s + (A + B + C),
\]
which gives us the system of equations
\[
B + D = 0, \quad A + B + C + 2D = 0, \quad B + 2C + D = 1, \quad A + B + C = 0.
\]
The solution of this system is \(A = -1/2, B = 0, C = 1/2, D = 0\), which means that
\[
Y(s) = \frac{1}{s + 1} + \frac{1}{(s + 1)^2} + e^{-\pi s/2} \left( -\frac{1/2}{(s + 1)^2} + \frac{1/2}{s^2 + 1} \right)
\]
Therefore
\[ y(t) = e^{-t} + te^{-t} + u_{-\pi/2}(t) \left( -\frac{1}{2}(t - \pi/2)e^{-(t-\pi/2)} + \frac{1}{2}\sin(t - \pi/2) \right). \]

(b) First, we use formula 19 from page 304 of the text:
\[
\mathcal{L}\{te^{2t}\sin(3t)\} = (-1)^1 \frac{d}{ds} \mathcal{L}\{e^{2t}\sin(3t)\}.
\]
Then formula 9 gives us
\[
\mathcal{L}\{e^{2t}\sin(3t)\} = \frac{3}{(s - 2)^2 + 3^2},
\]
so
\[
\mathcal{L}\{te^{2t}\sin(3t)\} = -\frac{d}{ds} \left( \frac{3}{s^2 - 4s + 13} \right) = \frac{3(2s - 4)}{(s^2 - 4s + 13)^2}.
\]

7. The eigenvalues are the solutions of the quadratic equation \((1 - r)(-2 - r) - 1 \cdot 4 = 0\) or \(r^2 + r - 6 = 0\), so \(r = 2, -3\). The eigenvector corresponding to \(r = 2\) is obtained from the matrix
\[
\begin{pmatrix}
-1 & 1 \\
4 & -4
\end{pmatrix},
\]
which means it’s
\[
\begin{pmatrix}
1 \\
1
\end{pmatrix}.
\]
The matrix for the eigenvalue \(r = -3\) is
\[
\begin{pmatrix}
4 & 1 \\
4 & 1
\end{pmatrix},
\]
which means this eigenvector is
\[
\begin{pmatrix}
1 \\
-4
\end{pmatrix},
\]
the general solution of the differential equation is
\[
y = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}.
\]

8. The differential equation can be written as
\[
y'' = (x + 2)y
\]
so \(y''(0) = 2\). Differentiation gives
\[
y''' = y + (x + 2)y'
\]
so \(y'''(0) = 1 + 2 = 3\) and then
\[
y^{(4)} = 2y' + (x + 2)y'',
\]
so \(y^{(4)}(0) = 2(1) + 2(2) = 6\). The first five non-zero terms are
\[
y = 1 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{6}{4!}x^4 + \ldots.
\]
9. (a) There are four possibilities to check for critical points. The first is when $2 + x = 0$ and $4 - x = 0$, and this pair of equations has no solutions. The second is $2 + x = 0$ and $y + x = 0$, which gives $(-2, 2)$. The third is $y - x = 0$ and $-x = 0$, which gives $(4, 4)$. The last one is $y - x = 0$ and $y + x = 0$, which gives $(0, 0)$.

(b) (Notice that $(4, 4)$ was listed in part (a).) To find the corresponding linear system, we compute the partial derivatives

$$\frac{\partial}{\partial x}((2 + x)(y - x)) = \frac{\partial}{\partial x}(2y - 2x + xy - x^2) = -2 + y - 2x,$$

$$\frac{\partial}{\partial x}((4 - x)(y + x)) = \frac{\partial}{\partial x}(4y - 4x - xy - x^2) = 4 - y - 2x,$$

$$\frac{\partial}{\partial y}((2 + x)(y - x)) = 2 + x,$$

$$\frac{\partial}{\partial y}(4 - x)(y + x)) = 4 - x.$$

Therefore the linear system is

$$\mathbf{x}' = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix} \mathbf{x}.$$

(c) To determine the type and stability, we compute the eigenvalues of the matrix from part (b). We get the eigenvalues from the equation

$$0 = (-6 - r)(-r) - (-8)6 = r^2 + 6r + 48,$$

so the eigenvalues are

$$\frac{-6 \pm \sqrt{36 - 4(48)}}{2} = -3 \pm i\sqrt{39}.$$

They are complex with negative real part, so the critical point is an asymptotically stable spiral point.