Math 267, Section A1
Professor Lieberman
May 7, 2002

SOLUTIONS FOR PRACTICE FINAL

Corrected on May 9.

1. (Not for Spring, 2002)

2. First, we solve the homogeneous equation

$$y_h'' - 2y_h' - 3y_h = 0.$$

The solution has the form $y_h = e^{rt}$ with $r^2 - 2r - 3 = 0$ so $r = 3, -1$. Then the solution of the original equation has the form

$$y = u_1 e^{3t} + u_2 e^{-t}$$

for functions u_1 and u_2 which solve the system of equations

$$u_1' e^{3t} + u_2' e^{-t} = 0$$
$$u_1' 3e^{3t} + u_2'(-1)e^{-t} = 3e^{2t}.$$

Add these two equations to see that $4u_1' e^{3t} = 3e^{2t}$ or $u_1' = (3/4)e^{-t}$. Plugging into the first equation for u_1' and u_2' gives $u_2' = (-3/4)u^{3t}$, so

$$y = -\frac{3}{4}e^{2t} + c_1 e^{3t} + -\frac{1}{4}e^{2t} + c_2 e^{-t}.$$

3. This equation is exact, so the solution has the form $\psi(x, y) = C$ with

$$\psi(x, y) = \int (2xy^2 + 2y)\, dx = x^2y^2 + 2xy + f(y),$$

and

$$\psi(x, y) = \int (2x^2y + 2x)\, dy = x^2y^2 + 2xy + g(x).$$

If we set these two expressions equal, we see that $f(y) = g(x)$, so f and g are constants, which we can take to be zero. Therefore $\psi(x, y) = x^2y^2 + 2xy$, and the solution of the differential equation is

$$x^2y^2 + 2xy = C.$$

4. The integrating factor μ solves the equation

$$\frac{\mu'}{\mu} = \cot x,$$

so $\ln \mu = \ln(\sin x)$, or $\mu = \sin x$. Therefore,

$$((\sin x)y)' = 2 \csc x \sin x = 2.$$

Integrating this equation gives $(\sin x)y = 2x + C$. From the initial condition $(\sin \frac{\pi}{2})1 = 2\frac{\pi}{2} + C$, which simplifies to $C = 1 - \pi$. Therefore

$$y = \frac{2x + 1 - \pi}{\sin x}.$$

5. The differential equation has as its solution $y = e^{rx}$ with r satisfying the equation $r^2 + 4r + 5 = 0$. Therefore $r = -2 \pm i$, and the general solution is

$$y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x.$$

Since

$$y' = -2c_1 e^{-2x} \cos x - c_1 e^{-2x} \sin x - 2c_2 e^{-2x} \sin x + c_2 e^{-2x} \cos x,$$

the initial conditions become $c_1 = 1$ and $-2c_1 + c_2 = 0$, so $c_2 = 2$ and the solution is

$$y = e^{-2x} \cos x + 2e^{-2x} \sin x.$$

6. (a) First, we rewrite f:

$$f(t) = u_{\pi/2}(t) \sin t = u_{\pi/2}(t)[\cos(t - \frac{\pi}{2})],$$

so

$$\mathcal{L}\{f(t)\} = e^{-\pi s/2} \frac{s}{s^2 + 1}.$$

Then the Laplace transform Y of the solution of the differential equation satisfies the algebraic equation

$$(s^2 Y(s) - s) + 2(sY(s) - 1) + Y(s) = e^{-\pi s/2} \frac{s}{s^2 + 1}$$

and therefore

$$Y(s) = \frac{s + 2}{s^2 + 2s + 1} + e^{-\pi s/2} \frac{s}{(s^2 + 2s + 1)(s^2 + 1)}.$$

Since $s^2 + 2s + 1 = (s + 1)^2$, we have

$$\frac{s + 2}{s^2 + 2s + 1} = \frac{1}{s + 1} + \frac{1}{(s + 1)^2},$$

and

$$\frac{s}{(s^2 + 2s + 1)(s^2 + 1)} = \frac{A}{(s + 1)^2} + \frac{B}{s + 1} + \frac{C}{s^2 + 1} + \frac{Ds}{s^2 + 1}.$$

The coefficients A, B, C, D satisfy the equation

$$s = A(s^2 + 1) + B(s + 1)(s^2 + 1) + C(s + 1)^2 + Ds(s + 1)^2,$$

or

$$s = (B + D)s^3 + (A + B + C + 2D)s^2 + (B + 2C + D)s + (A + B + C),$$

which gives us the system of equations

$$B + D = 0,\ A + B + C + 2D = 0,\ B + 2C + D = 1,\ A + B + C = 0.$$

The solution of this system is $A = -1/2$, $B = 0$, $C = 1/2$, $D = 0$, which means that

$$Y(s) = \frac{1}{s + 1} + \frac{1}{(s + 1)^2} + e^{-\pi s/2} \left(-\frac{1/2}{(s + 1)^2} + \frac{1/2}{s^2 + 1} \right)$$

Therefore

$$y(t) = e^{-t} + te^{-t} + u_{-\pi/2}(t)\left(-\frac{1}{2}(t-\pi/2)e^{-(t-\pi/2)} + \frac{1}{2}\sin(t-\pi/2)\right).$$

(b) First, we use formula 19 from page 304 of the text:

$$\mathcal{L}\{te^{2t}\sin(3t)\} = (-1)^1\frac{d}{ds}\mathcal{L}\{e^{2t}\sin(3t)\}.$$

Then formula 9 gives us

$$\mathcal{L}\{e^{2t}\sin(3t)\} = \frac{3}{(s-2)^2+3^2},$$

so

$$\mathcal{L}\{te^{2t}\sin(3t)\} = -\frac{d}{ds}\left(\frac{3}{s^2-4s+13}\right) = \frac{3(2s-4)}{(s^2-4s+13)^2}.$$

7. The eigenvalues are the solutions of the quadratic equation $(1-r)(-2-r) - 1\cdot 4 = 0$ or $r^2 + r - 6 = 0$, so $r = 2, -3$. The eigenvector corresponding to $r = 2$ is obtained from the matrix

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix},$$

which means it's

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The matrix for the eigenvalue $r = -3$ is

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix},$$

which means this eigenvector is

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

the general solution of the differential equation is

$$y = c_1e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-3t}\begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

8. The differential equation can be written as

$$y'' = (x+2)y$$

so $y''(0) = 2$. Differentiation gives

$$y''' = y + (x+2)y'$$

so $y'''(0) = 1 + 2 = 3$ and then

$$y^{(4)} = 2y' + (x+2)y'',$$

so $y^{(4)}(0) = 2(1) + 2(2) = 6$. The first five non-zero terms are

$$y = 1 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{6}{4!}x^4 + \ldots.$$

9. (a) There are four possibilities to check for critical points. The first is when $2 + x = 0$ and $4 - x = 0$, and this pair of equations has no solutions. The second is $2 + x = 0$ and $y + x = 0$, which gives $(-2, 2)$. The third is $y - x = 0$ and $-x = 0$, which gives $(4, 4)$. The last one is $y - x = 0$ and $y + x = 0$, which gives $(0, 0)$.

   (b) (Notice that $(4, 4)$ was listed in part (a).) To find the corresponding linear system, we compute the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x}((2+x)(y-x)) &= \frac{\partial}{\partial x}(2y - 2x + xy - x^2) = -2 + y - 2x, \\ \frac{\partial}{\partial x}((4-x)(y+x)) &= \frac{\partial}{\partial x}(4y - 4x - xy - x^2) = 4 - y - 2x, \\ \frac{\partial}{\partial y}((2+x)(y-x)) &= 2 + x, \\ \frac{\partial}{\partial y}(4-x)(y+x)) &= 4 - x. \end{aligned}$$

   Therefore the linear system is

$$\mathbf{x}' = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix} \mathbf{x}.$$

   (c) To determine the type and stability, we compute the eigenvalues of the matrix from part (b). We get the eigenvalues from the equation

$$0 = (-6 - r)(-r) - (-8)6 = r^2 + 6r + 48,$$

   so the eigenvalues are

$$\frac{-6 \pm \sqrt{36 - 4(48)}}{2} = -3 \pm i\sqrt{39}.$$

   They are complex with negative real part, so the critical point is an asymptotically stable spiral point.