Math 165, Section D
Professor Lieberman

QUIZ #5

Question:
The fraction

$$\frac{19}{95}$$

has the cancellation property

$$\frac{19}{95} = \frac{1\not{9}}{\not{9}5} = \frac{1}{5}.$$

Using pencil and paper, find all fractions with two-digit numerator and denominator that have this property. (Most of your score will based on showing your work.)

Answer: The correct form of the cancellation property is that

$$\frac{10x + y}{10y + z} = \frac{x}{z}.$$

If you cross multiply, you get $10xz + yz = 10xy + xz$, which simplifies to

$$(9x + y)z = 10xy.$$

Since the right hand side of this equation is divisible by 10, so is the left hand side. This leads to three possibilities:

(1) $9x + y$ is divisible by 2 and $z = 5$.
(2) $9x + y$ is divisible by 5 and z is even.
(3) $9x + y$ is divisible by 10.

In the first case, we have $9x + y = 2xy$. If $x = 1$, we have $9 + y = 2y$, so $y = 9$. This gives the cancellation from the question

$$\frac{19}{95} = \frac{1}{5}.$$

If $x \geq 2$, we rewrite $9x + y = 2xy$ as $y = (2y - 9)x$. Because $1 \leq y \leq 9$, it follows that $1/2 \leq 2y - 9 \leq 9/2$, and therefore (recalling that y is an integer) y must be 5 or 6. If $y = 5$, then $9x + 5 = 10x$, so $x = 5$, which gives

$$\frac{55}{55} = \frac{5}{5}.$$

If $y = 6$, then $9x + 6 = 12x$, so $x = 2$, which gives

$$\frac{26}{65} = \frac{2}{5}.$$

The second case is divided into four subcases: $z = 2$, $z = 4$, $z = 6$, and $z = 8$.

If $z = 2$, we have $9x + y = 5xy$. If $x = 1$, then this equation becomes $9 + y = 5y$, which does not have a whole number solution. If $x \geq 2$, then we rewrite our equation as

$y = (5y-9)x$. Therefore $1/2 \leq 5y-9 \leq 9/2$, so $19/10 \leq y \leq 27/10$, which means $y = 2$ and then $9x + 2 = 10x$, which means also $x = 2$. Therefore we obtain

$$\frac{22}{22} = \frac{2}{2}.$$

If $z = 4$, our equation is $9x + y = (5/2)xy$. If also $x = 1$, we find that $y = 6$, and we have

$$\frac{16}{64} = \frac{1}{4}.$$

If $x \geq 2$, then $1/2 \leq (5/2)y - 9 \leq 9/2$, so $y = 4$ or $y = 5$. If $y = 4$, then $x = 4$, so we obtain

$$\frac{44}{44} = \frac{4}{4}.$$

If $y = 5$, then the equation $9x + 5 = (25/2)x$ doesn't have a whole number solution.

If $z = 6$, our equation is $9x + y = (5/3)xy$. This equation doesn't have a whole number solution for $x = 1$, so we can directly consider $x \geq 2$, which means $1/2 \leq (5/3)y - 9 \leq 9/2$. Therefore $57/10 \leq y \leq 81/10$ or $y = 6, 7, 8$. If $y = 6$, then $x = 6$, which yields

$$\frac{66}{66} = \frac{6}{6}.$$

The corresponding equations for $y = 7$ and $y = 8$ do not have whole number solutions.

If $z = 8$, our equation is $9x + y = (5/4)xy$. The equation with $x = 1$ has the solution $y = 36$, which is too big, so we can go to $x \geq 2$. Then $1/2 \geq (5/4)y - 9 \leq 9/2$, so $y = 8$ or $y = 9$. If $y = 8$, then $x = 8$, which gives

$$\frac{88}{88} = \frac{8}{8}.$$

If $y = 9$, then $x = 4$, which gives

$$\frac{49}{98} = \frac{4}{8}.$$

Finally, if $9x + y$ is divisible by 10, so is $y - x$ (because $y - x = 9x + y - 10x$). Since x and y are single digits, it follows that $x = y$, and the original equation $(9x + y)z = 10xy$ tells us that also $z = x$. Some of these cases have already been listed. The others are:

$$\frac{11}{11} = \frac{1}{1}, \ \frac{33}{33} = \frac{3}{3}, \ \frac{77}{77} = \frac{7}{7}, \ \frac{99}{99} = \frac{9}{9}.$$

Therefore, there are 13 fractions with the cancellation property.