Small Ramsey Numbers

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**Definition**

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer $n$ such that any $k$-edge coloring of $K_n$ contains a copy of $G_i$ in color $i$ for some $1 \leq i \leq k$.

\[ R(K_3, K_3) > 5 \quad \text{and} \quad R(K_3, K_3) \leq 6 \]
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**Theorem (Ramsey 1930)**

\[ R(K_m, K_n) \text{ is finite.} \]
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Questions:

- study how \( R(G_1, \ldots, G_k) \) grows if \( G_1, \ldots, G_k \) grow (large)
- study \( R(G_1, \ldots, G_k) \) for fixed \( G_1, \ldots, G_k \) (small)
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Radziszowski - *Small Ramsey Numbers*

Electronic Journal of Combinatorics - Survey
Seminal paper:
David P. Robbins Prize by AMS for Razborov in 2013
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**Example (Goodman, Razborov)**
If the density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- designed to attack extremal problems.
- works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs)
- the results are in limit (very large graphs)
## Applications (Incomplete List)

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<td>Razborov</td>
<td>2008</td>
<td>edge density vs. triangle density</td>
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<td>Hladký, Král', Norin</td>
<td>2009</td>
<td>Bounds for the Caccetta-Haggvist conjecture</td>
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<td>Razborov</td>
<td>2010</td>
<td>3-hypergraphs with forbidden 4-vertex configurations</td>
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<td>Hatami, Hladký, Král', Norine, Razborov / Grzesik</td>
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<td>Hatami, Hladký, Král', Norin, Razborov</td>
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<td>Non-Three-Colourable Common Graphs Exist</td>
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<td>Balogh, Hu, Lidický, Liu / Baber</td>
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<td>4-cycles in hypercubes</td>
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<td>Reiher</td>
<td>2012</td>
<td>edge density vs. clique density</td>
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<td>Das, Huang, Ma, Naves, Sudakov</td>
<td>2013</td>
<td>minimum number of $k$-cliques</td>
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<td>Baber, Talbot</td>
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<td>A Solution to the 2/3 Conjecture</td>
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<td>Falgas-Ravry, Vaughan</td>
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<td>Turán density of many 3-graphs</td>
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<td>Cummings, Král', Pfender, Sperfeld, Treglown, Young</td>
<td>2013</td>
<td>Monochromatic triangles in 3-edge colored graphs</td>
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<td>Kramer, Martin, Young</td>
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<td>Boolean lattice</td>
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<td>Balogh, Hu, Lidický, Pikhurko, Udvari, Volec</td>
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<td>Monotone permutations</td>
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<td>Norin, Zwols</td>
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<td>New bound on Zarankiewicz's conjecture</td>
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<td>Huang, Linial, Naves, Peled, Sudakov</td>
<td>2014</td>
<td>3-local profiles of graphs</td>
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<td>Rainbow triangles in 3-edge colored graphs</td>
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<tr>
<td>Balogh, Hu, Lidický, Pfender</td>
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<td>Induced density of $C_5$</td>
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<td>Goaoc, Hubard, de Verclos, Séréni, Volec</td>
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<td>Order type and density of convex subsets</td>
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<tr>
<td>Coregliano, Razborov</td>
<td>2015</td>
<td>Tournaments</td>
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</tbody>
</table>

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, geometry, ... Razborov: Flag Algebra: an Interim Report
Inspiration

Theorem (Cummings, Král, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on $n$ vertices, there are at least $\frac{1}{25} \binom{n}{3} + o(n^3)$ monochromatic triangles.

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\begin{align*}
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What is number of non-edges in a blow-up?
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\sum_{i=1}^{5} \binom{|I_i|}{2} \geq \sum_{i=1}^{5} \binom{n/5}{2} \geq 5 \binom{n/5}{2} \approx \frac{1}{5} \binom{n}{2}
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What is number of non-edges in a blow-up?

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**Observation (Key Observation)**

If a Ramsey graph $G$ has $k$ vertices, then the density of non-edges in any blow-up of $G$ is at least $\frac{1}{k} + o(1)$. 
Outline of idea

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- Let $G$ be 2-edge-colored complete graphs with no monochromatic triangle.
- Consider all blow-ups $B$ of graphs in $G$.
- $\forall B \in B$, density of non-edges in $B$ is at least $\frac{1}{k} = \frac{1}{5}$.
Outline of Idea

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If the density of non-edges $\rho$ is $> \frac{1}{k+1}$ over all $B \in \mathcal{B}$, then a Ramsey graph has at most $k$ vertices.
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*If the density of non-edges $\rho$ is $> \frac{1}{k+1}$ over all $B \in B$, then a Ramsey graph has as most $k$ vertices.*

If one can prove $\rho > \frac{1}{6}$, then there is no Ramsey graph on 6 vertices.
Outline of idea

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If a Ramsey graph \( G \) has \( k \) vertices, then the density of non-edges in any blow-up of \( G \) is at least \( \frac{1}{k} + o(1) \).

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Notice that any lower bound on \( \rho \) in \( \left( \frac{1}{k+1}, \frac{1}{k} \right] \) gives that any Ramsey graph has at most \( k \) vertices.
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Notice that any lower bound on $\rho$ in $(\frac{1}{k+1}, \frac{1}{k}]$ gives that any Ramsey graph has at most $k$ vertices.
How can we characterize blow-ups $\mathcal{B}$ of graphs with no $\begin{array}{c} \text{I}_1 \\ \text{I}_2 \end{array}$, $\begin{array}{c} \text{I}_3 \\ \text{I}_4 \end{array}$?
**Blow-ups in Flag Algebra**

How can we characterize blow-ups $\mathcal{B}$ of graphs with no $\begin{array}{c} \text{Forbidden subgraphs:} \\
\begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array}
\end{array}
\end{array}$?
How can we characterize blow-ups $\mathcal{B}$ of graphs with no $\Delta$, $\Delta$?

Forbidden subgraphs:

\[ \text{minimize} \quad \sum_{i=1}^{5} I_i \quad \text{subject to} \quad \sum_{i=1}^{5} I_i = 0 \]
**Blow-ups in Flag Algebra**

How can we characterize blow-ups $\mathcal{B}$ of graphs with no forbidden subgraphs: $\begin{array}{c}
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\text{subject to }
\end{array}
\end{array}$

Flag Algebra question! Easy to modify.
### New upper bounds (so far)

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<tr>
<th>Problem</th>
<th>Lower</th>
<th>New upper</th>
<th>Old upper</th>
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<td>( R(K_4^-, K_8^-) )</td>
<td>29</td>
<td>32</td>
<td>38</td>
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<td>( R(K_4^-, K_9^-) )</td>
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<td>( R(K_4, K_7^-) )</td>
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<td>( R(K_9, C_5) )</td>
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<td>$R(K_4^-, K_5^-; 3)$</td>
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$R(K_4^-, K_5^-; 3) = 12$

- No lower bound better than 10 was known.
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  ...and suggest that a Ramsey graph on 11 vertices can only have subgraphs on 8 vertices from a very short list.
- Generating all graphs from this short list is not difficult, and the (unique?) Ramsey graph can be found.
**Example of Computation**

**Lemma**

\[ R(K_3, K_3) \leq 6 \]
**Example of Computation**

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\[ R(K_3, K_3) \leq 6 \]

Our goal is to show:

\[
\begin{align*}
\frac{1}{6} & > \text{subject to} \quad \begin{array}{c}
\text{triangle} \\
\text{triangle} \\
\text{triangle} \\
\text{triangle} \\
\text{triangle} \\
\text{triangle}
\end{array}
\end{align*}
\]
**Lemma**

\[ R(K_3, K_3) \leq 6 \]

Our goal is to show:

\[ \frac{1}{6} \text{ subject to } \begin{array}{c}
\bullet \\
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\bullet \\
\end{array} = \begin{array}{c}
\bullet \\
\bullet \\
\end{array} = 0 \]

We show perhaps the most complicated proof of the lemma!
Our goal is to show:

\( \frac{1}{6} \) subject to

\[
\begin{align*}
\begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
\end{align*}
\]

\( = = = = = 0 \)
Our goal is to show:

$$\frac{1}{6} \text{ subject to } \begin{align*}
\text{triangle} &= \text{triangle} = \text{V} = \text{V} = 0
\end{align*}$$

Observe that \begin{align*}
\text{red line} \quad \text{and} \quad \text{blue line}
\end{align*} can be swapped.
Our goal is to show:

\[ \frac{1}{6} > \text{subject to} \]

\[ \begin{array}{ccccccc}
\text{\textbullet} & 1 \\
\end{array} \]

Observe that \[ \begin{array}{ccccccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

and \[ \begin{array}{ccccccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

can be swapped. Change to a color-blind setting. \[ \begin{array}{ccccccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

is a monochromatic triangle (red or blue).
Our goal is to show:

\[ \frac{1}{6} > \quad \text{subject to} \quad \begin{array}{c}
\end{array} \]

Observe that these and can be swapped. Change to a color-blind setting. is a monochromatic triangle (red or blue).

Our new goal is to show:

\[ \frac{1}{6} > \quad \text{subject to} \quad \begin{array}{c}
\end{array} \]
Our goal is to show:

\[ \frac{1}{6} > \text{subject to} \left\{ \begin{array}{l}
\bigcirc
\end{array} \right. = 0 \]

Observe that \( \bigcirc \) and \( \bigcirc \) can be swapped. Change to a color-blind setting. \( \bigcirc \) is a monochromatic triangle (red or blue).

Our new goal is to show:

\[ \frac{1}{6} > \text{subject to} \left\{ \begin{array}{l}
\bigcirc
\end{array} \right. = 0 \]

Color-blind setting will allow us to fit the computation on these slides.
Also important for bigger applications.
Our goal is to show:

\[ \frac{1}{6} > \text{subject to } \begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array} = \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} = \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} = 0 \]
Our goal is to show:

\[
\frac{1}{6} > 0 
\]

subject to

\[
\begin{align*}
\triangle &= \gamma = \square = 0 \\

\end{align*}
\]

Basic equations:

\[
\begin{align*}
\cdot &+ \cdot + \cdot + \cdot + \cdot + \cdot + \cdot = 1 \\

\end{align*}
\]

\[
= \frac{1}{6} \left( 1 + 0 + 0 + 1 + 3 + 2 + 6 \right)
\]
We use flags with type $\sigma_1$ of size two

$$F = \begin{pmatrix} \mathcal{V}, & \mathcal{V}, & \mathcal{V} \end{pmatrix}^T.$$  

For a positive semidefinite matrix $M$

$$0 \leq \left[ F^T M F \right]_{\sigma_1} = \left[ F^T \begin{pmatrix} 0.0744 & -0.0223 & -0.0520 \\ -0.0223 & 0.0238 & -0.0014 \\ -0.0520 & -0.0014 & 0.0536 \end{pmatrix} F \right]_{\sigma_1}$$

$$= -0.0116 \mathcal{V} - 0.3568 \mathcal{X} - 0.1784 \mathcal{X} - 0.0112 \mathcal{X}$$

$$+ 0.3216 \mathcal{V} + 0 \mathcal{X} + 0 \mathcal{X}.$$  

$$\left[ \begin{pmatrix} \mathcal{V} \times \mathcal{V} \end{pmatrix} \right]_{\sigma_1} = \left[ \frac{1}{2} \mathcal{V} + \frac{1}{2} \mathcal{V} \right]_{\sigma_1} = \frac{1}{2} \left( \frac{8}{12} \mathcal{X} + \frac{4}{12} \mathcal{X} \right).$$
\[ \frac{1}{6} \left( 1 + 0 + 0 + 1 + 3 + 2 + 6 \right) \]

We sum the equations and obtain

\[ 0 \geq 0.0116 + 0.3568 + 0.1784 + 0.0112 \]

\[ -0.3216 + 0 + 0 \]
\[
\frac{1}{6} \left( 1 + 0 + 0 + 1 + 3 + 2 + 6 \right) = 0.1782 + 0.3568 + 0.1784 + 0.1778
\]

We sum the equations and obtain

\[
0 \geq 0.0116 + 0.3568 + 0.1784 + 0.0112
\]

\[-0.3216 + 0 + 0 \geq 0.1782 + 0.3568 + 0.1784 + 0.1778
\]

\[+ 0.1784 + 0.33 \geq 0.1782 + 0.3568 + 0.1784 + 0.1778\]
\[ \frac{1}{6} \left( \begin{array}{cccccc} 1 & + & 0 & + & 0 & + 1 & + 3 & + 2 & + 6 \end{array} \right) \]

\[ 0 \geq 0.0116 + 0.3568 + 0.1784 + 0.0112 \]

\[ -0.3216 + 0 + 0 \]

We sum the equations and obtain

\[ \geq 0.1782 + 0.3568 + 0.1784 + 0.1778 \]

\[ + 0.1784 + 0.33 + \]

\[ \geq 0.17 \left( \begin{array}{cccccc} + & + & + & + & + & + \end{array} \right) \]
\[
\sum = \frac{1}{6} \left( 1 + 0 + 0 + 1 + 3 + 2 + 6 \right)
\]

\[
0 \geq 0.0116 + 0.3568 + 0.1784 + 0.0112
\]

\[
-0.3216 + 0 + 0
\]

We sum the equations and obtain

\[
\sum \geq 0.1782 + 0.3568 + 0.1784 + 0.1778
\]

\[
+ 0.1784 + 0.33 +
\]

\[
\geq 0.17 \left( + + + + + + + \right)
\]

\[
> 0.17 > \frac{1}{6}.
\]
We proved

\[ \begin{align*}
\bullet > \frac{1}{6} & \text{ subject to } \\
\bullet & \text{ } = \text{ } = \text{ } = \text{ } = \text{ } = 0
\end{align*} \]
We proved
\[
\begin{array}{c}
> \frac{1}{6} \\
\end{array}
\]
subject to
\[
\begin{array}{c}
= \\
= \\
= \\
= \\
= \\
= 0
\end{array}
\]
Hence Ramsey graph for \(K_3\) and \(K_3\) has less than 6 vertices.
And therefore \(R(K_3, K_3) \leq 6\).
We proved

\[ \frac{1}{6} > \text{subject to } \begin{array}{c}
\text{triangle} = \text{triangle} = \text{V} = \text{V} = \text{V} = \text{V} = 0
\end{array} \]

Hence Ramsey graph for \( K_3 \) and \( K_3 \) has less than 6 vertices.
And therefore \( R(K_3, K_3) \leq 6 \).

Note that the matrix \( M \) was not unique or tight (easy rounding).
(bound \( \geq \frac{1}{5} \) is also obtainable)
We proved

\[ \frac{1}{6} > \frac{1}{6} \text{ subject to } \begin{align*}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix} = 0
\end{align*} \]

Hence Ramsey graph for \( \triangle \) and \( \triangle \) has less than 6 vertices.
And therefore \( R(K_3, K_3) \leq 6 \).

Note that the matrix \( M \) was not unique or tight (easy rounding).
(bound \( \geq \frac{1}{5} \) is also obtainable)

Thank you for your attention!