# Short proofs of coloring theorems on planar graphs

Oleg V. Borodin, Alexandr V. Kostochka, Bernard Lidický, Matthew Yancey

Sobolev Institute of Mathematics and Novosibirsk State University
University of Illinois at Urbana-Champaign

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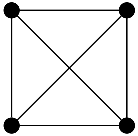
# Definitions (4-critical graphs)

graph $G = (V, E)$

*coloring* is $\varphi : V \to C$ such that $\varphi(u) \neq \varphi(v)$ if $uv \in E$

$G$ is a *$k$-colorable* if coloring with $|C| = k$ exists

$G$ is a *4-critical graph* if $G$ is not 3-colorable
but every $H \subset G$ is 3-colorable.



# Inspiration

**Theorem (Grötzsch '59)**
*Every planar triangle-free graph is 3-colorable.*

Recently reproved by Kostochka and Yancey using

**Theorem (Kostochka and Yancey '12)**
*If $G$ is a 4-critical graph, then*

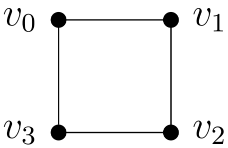
$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

used as $3|E(G)| \geq 5|V(G)| - 2$

# Every planar triangle-free graph is 3-colorable

Let $G$ be a minimal counterexample - not 3-colorable triangle-free plane graph, but every proper subgraph is. i.e. $G$ is 4-critical

CASE1 $G$ contains a 4-face (try 3-color $G$)

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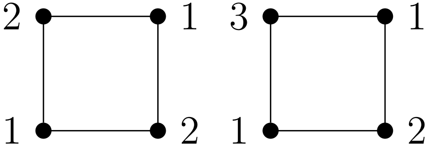


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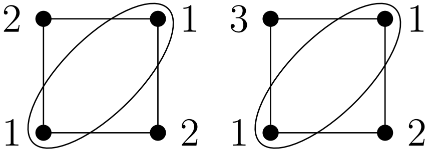


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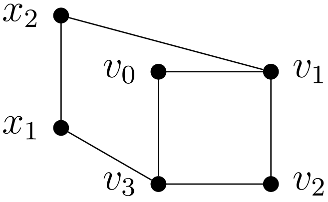


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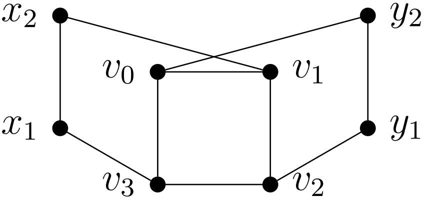


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# Every planar triangle-free graph is 3-colorable

Let $G$ be a minimal counterexample - not $3$-colorable triangle-free plane graph, but every proper subgraph is. i.e. $G$ is $4$-critical

CASE1 $G$ contains a $4$-face (try $3$-color $G$)

CASE2 $G$ contains no $4$-faces

$|E(G)| = e, |V(G)| = v, |F(G)| = f$.

- $v - 2 + f = e$ by Euler's formula
- $2e \geq 5f$ since face is at least a $5$-face
- $5v - 10 + 5f = 5e$
- $5v - 10 + 2e \geq 5e$
- $5v - 10 \geq 3e$ (our case)
- $3e \geq 5v - 2$ (every $4$-critical graph)

# Generalizations?

Theorem (Grötzsch '59)

*Every planar triangle-free graph is 3-colorable.*

Can be strengthened?

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**Theorem (Grötzsch '59)**

*Every planar triangle-free graph is 3-colorable.*

Can be strengthened?

Yes! - recall that CASE2

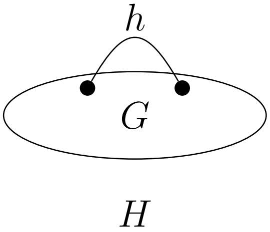
- $5v - 10 \geq 3e$ (no 3-,4-faces)
- $3e \geq 5v - 2$ (every 4-critical graph)

has some gap.

# Adding a bit

**Theorem (Aksenov '77; Jensen and Thomassen '00)**

*Let $G$ be a triangle-free planar graph and $H$ be a graph such that $G = H - h$ for some edge $h$ of $H$. Then $H$ is $3$-colorable.*



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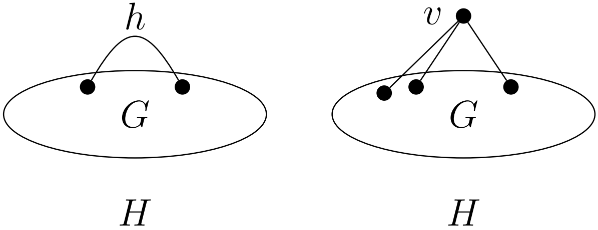
**Theorem (Jensen and Thomassen '00)**

*Let $G$ be a triangle-free planar graph and $H$ be a graph such that $G = H - v$ for some vertex $v$ of degree $3$. Then $H$ is $3$-colorable.*



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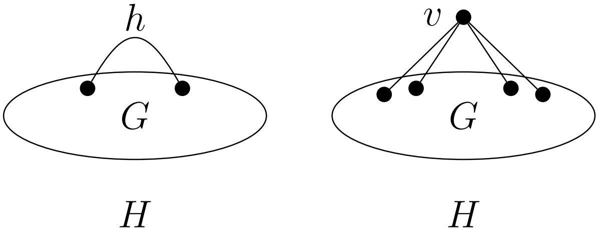

# For proof

## Theorem

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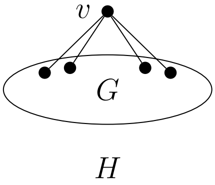
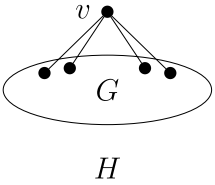


# Proof

$G$ plane, triangle-free, $G = H - v$,
$H$ is 4-critical




CASE1: No 4-faces in $G$

$V(H) = v, E(H) = e, V(G) = v - 1, E(G) = e - 4, F(G) = f$

- $5f \leq 2(e - 4)$ since $G$ has no 4-faces
- $(v - 1) + f - (e - 4) = 2$ by Euler's formula
- $5v + 5f - 5e = -5$
- $5v - 3e - 8 \geq -5$
- $5v - 3 \geq 3e$ (our case)
- but $3e \geq 5v - 2$ ($H$ is 4-criticality)

CASE2: 4-face $(v_0, v_1, v_2, v_3) \in G$

# Proof

$G$ plane, triangle-free, $G = H - v$,
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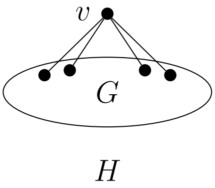
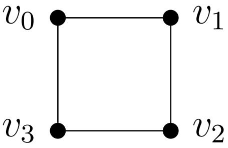



CASE1: No $4$-faces in $G$
CASE2: $4$-face $(v_0, v_1, v_2, v_3) \in G$



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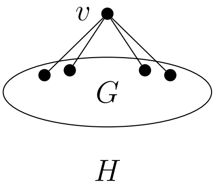


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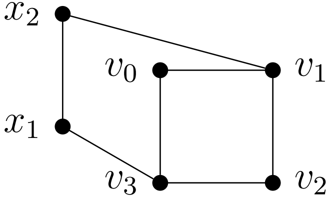

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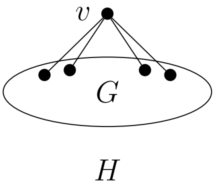


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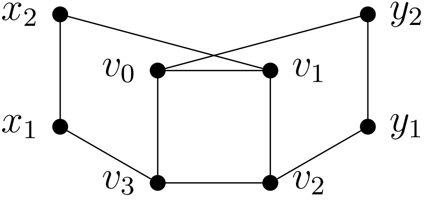
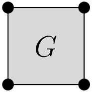
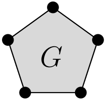


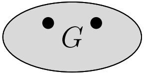
# Precoloring

**Theorem (Grötzsch '59)**

*Let $G$ be a triangle-free plane graph and $F$ be a face of $G$ of length at most $5$. Then each $3$-coloring of $F$ can be extended to a $3$-coloring of $G$.*




**Theorem (Aksenov et al. '02)**

*Let $G$ be a triangle-free planar graph. Then each coloring of two non-adjacent vertices can be extended to a $3$-coloring of $G$.*

# For proof

Theorem (Grötzsch '59)

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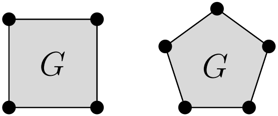
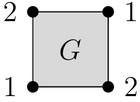


# Proof

If $G$ is a triangle-free plane graph, $F$ is a precolored 4-face or 5-face, then precoloring of $F$ extends.

CASE1: $F$ is a 4-face




CASE2: $F$ is a 5-face

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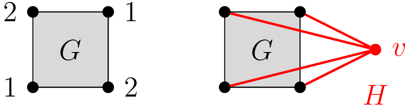


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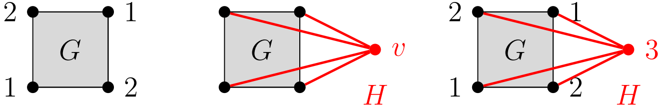


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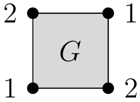


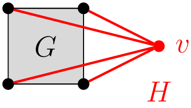


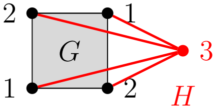


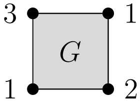


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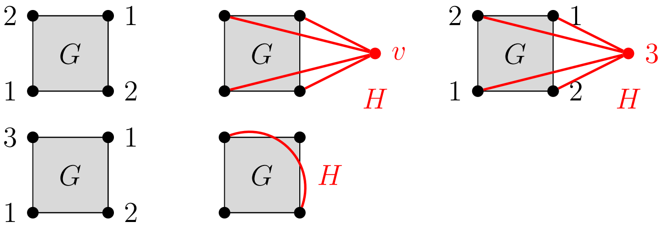


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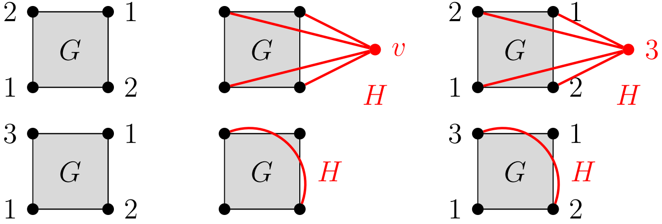


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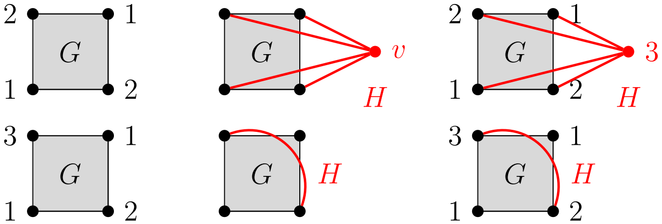


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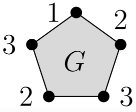

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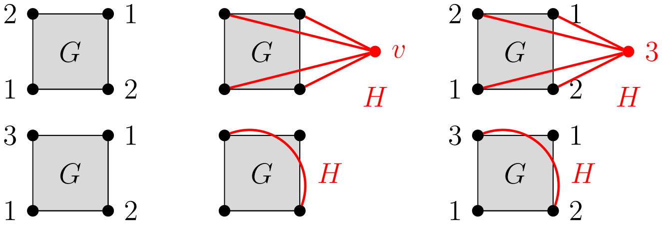


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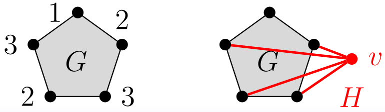

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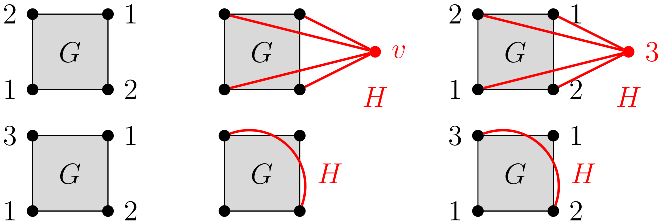


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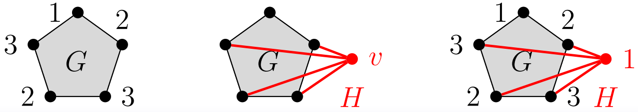

# Some triangles?

**Theorem (Grötzsch '59)**

*Every planar triangle-free graph is 3-colorable.*

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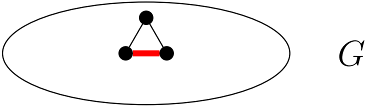
Theorem (Grötzsch '59)

*Every planar triangle-free graph is 3-colorable.*

We already showed one triangle!




Removing one edge of triangle results in triangle-free $G$.

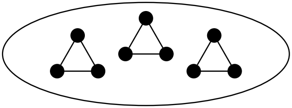
# Some triangles?

Theorem (Grötzsch ’59)

*Every planar triangle-free graph is 3-colorable.*

Theorem (Grünbaum ’63; Aksenov ’74; Borodin ’97)

*Let G be a planar graph containing at most three triangles. Then G is 3-colorable.*



$G$

# Three triangles - Proof outline

Theorem (Grünbaum ’63; Aksenov ’74; Borodin ’97)

*Let $G$ be a planar graph containing at most three triangles. Then $G$ is 3-colorable.*

- $G$ is 4-critical (minimal counterexample)
- 3-cycle is a face
- 4-cycle is a face or has a triangle inside and outside
- 5-cycle is a face or has a triangle inside and outside

CASE1: $G$ has no 4-faces
CASE2: $G$ has a 4-faces with triangle (no identification)
CASE3: $G$ has a 4-face where identification is possible

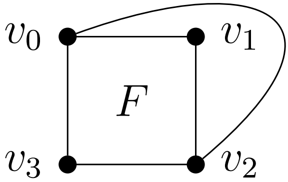
# Three triangles - Proof outline

CASE1: $G$ has no $4$-faces

- $v - 2 + f = e$ by Euler's formula
- $5v - 4 + 5f - 6 = 5e$
- $2e \geq 5(f-3) + 3 \cdot 3 = 5f - 6$ since $3$ triangles
- $5v - 4 \geq 3e$ (our case)
- $3e \geq 5v - 2$ (every $4$-critical graph)

# Three triangles - Proof outline

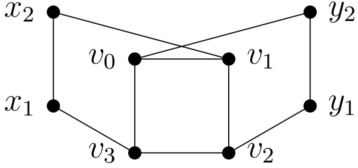
CASE2: $G$ has a 4-face $F$ with a triangle (no identification)




Both $v_0, v_1, v_2$ and $v_0, v_2, v_3$ are faces. $G$ has 4 vertices!

# Three triangles - Proof

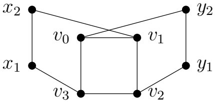
CASE3: $G$ has a $4$-face where identification is possible




Since $G$ is plane, some vertices are the same.

# Three triangles - Proof

CASE3: $G$ has a $4$-face where identification is possible

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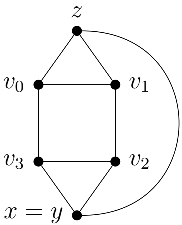


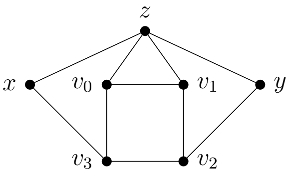

# Three triangles - Proof

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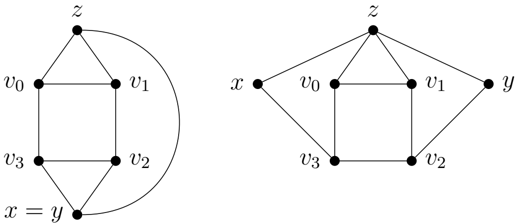
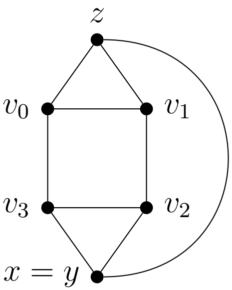



Only two cases left . . .

# One of the two cases left



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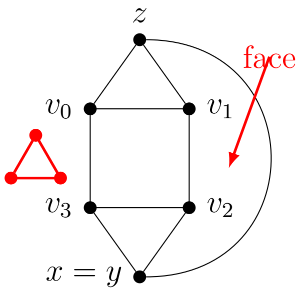

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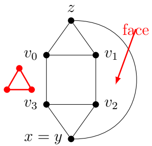


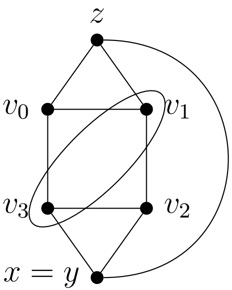

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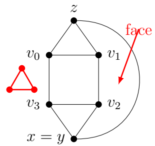


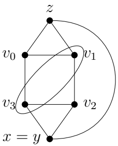


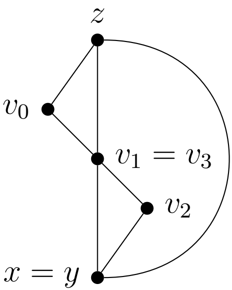

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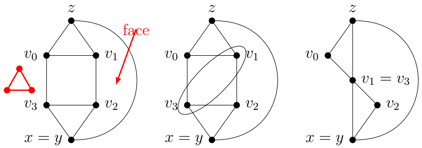


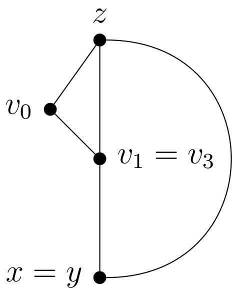

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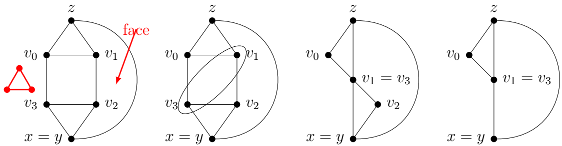


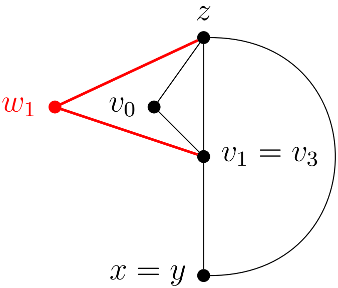

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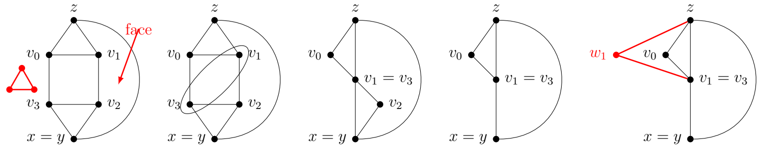


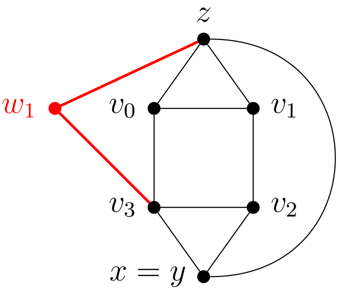

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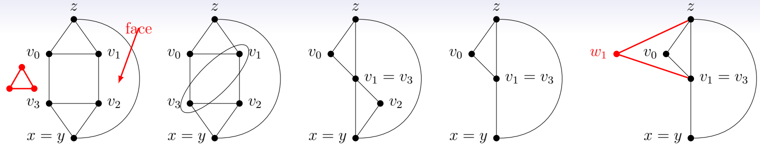


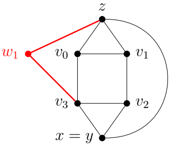


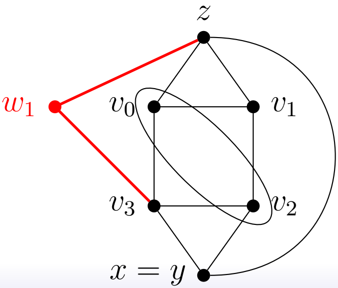

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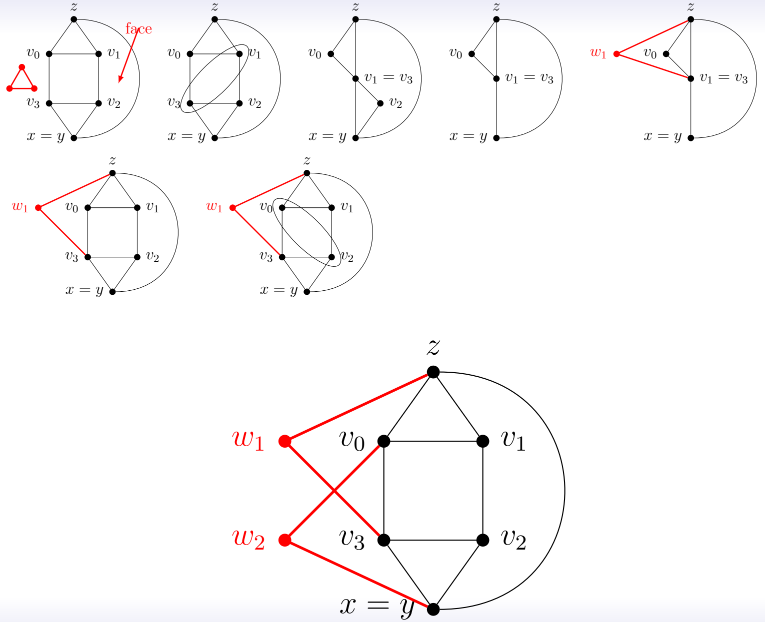

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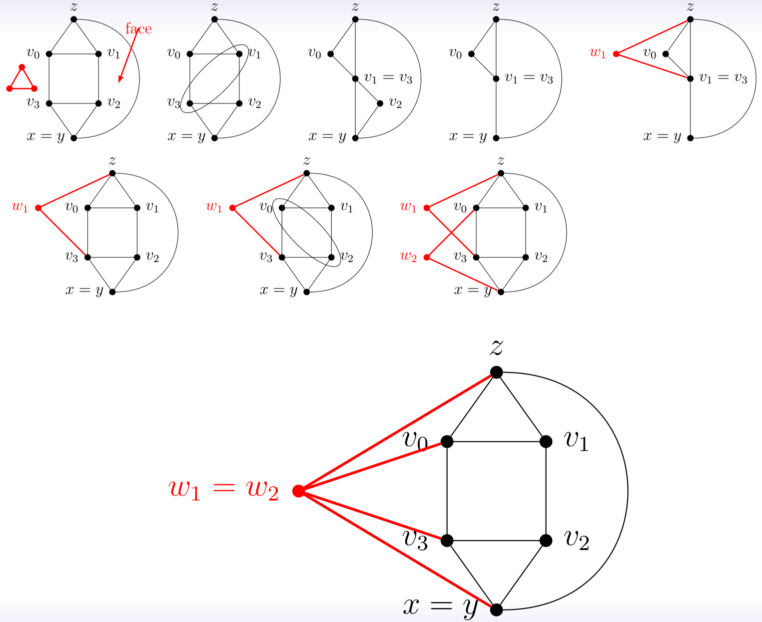

Thank you for your attention!

Slides available at
http://www.math.uiuc.edu/~lidicky/